MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XXIV.

MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

VICE-PRINCIPAL OF HUDDERSFIELD COLLEGE.

VOL. XXIV.

FROM JULY TO DECEMBER, 1875.

LONDON:

C. F. HODGSON & SON, GOUGH SQUARE,

FLEET STREET.

1876.

CORRIGENDA.

VOL. XXIII.

Page 81, line 18, *for* dx *read* $d\left\{x+(2Ry-y^2)^{\frac{1}{2}}\right\}$.

Page 82, line 28, *for* $(16)^3$ *read* $(14)^3$.

Page 83, line 5, *for* $(14)^6$ *read* $(14)^3$.

Page 90, line 25, *read* $M=\int_0^c M_1 \cdot \frac{x}{c} \cdot \frac{bx}{c} \cdot \frac{x^2}{c^2} \Delta\, dx \div \int_0^c \frac{bx}{c} \cdot \frac{x^2}{c^2} \Delta\, dx = \frac{4}{5} M_1$,

and then the result in this Solution will agree with Mr. TEBAY'S.

VOL. XXIV.

Page 46, line 24, *read* $B=-\frac{2q}{pr}$.

LIST OF CONTRIBUTORS.

ALDIS, J. S., M.A.; H.M. Inspector of Schools.
ALLMAN, GEO. S., LL.D.; Professor of Mathematics in the Queen's University, Galway.
BALL, ROBT. STAWELL, LL.D.; Astronomer Royal of Ireland, and Prof. of Astronomy in the Univ. of Dublin.
BATES, GILL; Prescott, Iowa, United States.
BATTAGLINI, GIUSEPPE; Professore di Matematiche nell' Università di Roma.
BILLS, SAMUEL; Newark-on-Trent.
BIRCH, Rev. J. G., M.A.; R. M. A., Woolwich.
BLACKWOOD, ELIZABETH; New York.
BLISSARD, Rev. J., B.A.; The Vicarage, Hampstead Norris, Berks.
BOOTH, Rev. Dr., F.R.S.; The Vicarage, Stone, near Aylesbury, Bucks.
BOURKE, Hon. ALGERNON; London.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BURNSIDE, W. S., M.A.; Trinity College, Dublin.
CAMPBELL, Capt. FRED.; Notting Hill, London.
CARR, G. S.; Clapham.
CARNT, E. C.; Royal Hospital School, Greenwich.
CASEY, JOHN, LL.D., F.R.S.; Prof. of Higher Mathematics in the Catholic Univ. of Ireland.
CAVALLIN, C. B. S.; University of Upsala.
CAYLEY, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge; Member of the Institute of France, &c.
CHADWICK, W.; Fellow of C. C. Coll., Oxford.
CLARKE, Colonel, C.B., F.R.S.; Ordnance Survey Office, Southampton.
CLARKE, MARY A.; Birmingham.
CLIFFORD, W. K., M.A.; Fellow of Trinity College, Cambridge; Professor of Applied Mathematics in University College, London.
COCKLE, Hon. Sir J., Knt., M.A., F.R.S.; Chief Justice of Queensland; President of the Queensland Philosophical Society.
COHEN, ARTHUR, M.A.; London.
COLLINS, JOHN, B.A.; Marlboro' Street, Dublin.
COLLINS, MATTHEW, LL.D.; Eden Quay, Dublin.
COLQUHOUN, MILLICENT; Inverness.
COLSON, C. G., M.A.; United Coll., St. Andrews.
CONOLLY, E.; Mountnugent, Ireland.
CORNER, S.; Wellington, Somerset.
COTTERILL, J. H. M.A.; Royal School of Naval Architecture, South Kensington.
COTTERILL, THOS., M.A.; London, late Fellow of St. John's College, Cambridge.
CREMONA, LUIGI; Direttore della Scuola degli Ingegneri, S. Pietro in Vinc di, Rome.
CROFTON, M. W., B.A., F.R.S.; Professor of Mathematics and Mechanics in the Royal Military Academy, Woolwich.
DALE, JAMES; Kingsland Place, Aberdeen.
DAVIS, JOHN; Clonakilty, Cork.
DAVIS, REUBEN; Bradford, Illinois.
DAVIS, R. F., B.A.; London.
DAVIS, WILLIAM BARRETT, B.A. London.
DINNIS, F. H., M.A.; Westbourne Ter., London.
DOBSON, T., B.A.; Head Master of Hexham Grammar School.
DRINKWATER, A.; London.
DUPAIN, J. C.; Professeur au Lycée d'Angoulême.
DYER, J. M., B.A.; Worcester Coll., Oxford.
EASTERBY, W., B.A.; Grammar School, St. Asaph.
EASTON, BELLE; Lockport, New York.
EDMUNDSON, GEORGE; Brasenose Coll., Oxford.
ELLIOTT, E.B., B.A.; Fellow of Queen's Coll., Oxon.
ESCOTT, ALBERT, M.A.; Head Master of the Royal Hospital School, Greenwich.
EVANS, ASHER B., M.A.; Lockport, New York.
EVERETT, J. D., D.C.L.; Professor of Natural Philosophy in the Queen's University, Belfast.
FICKLIN, JOSEPH; Professor of Mathematics and Astronomy in the University of Missouri.
FORDE, S., M.A.; Oxford.
FORTEY, H., M.A.; Bellary, Madras.
FRY, Colonel JOHN H.; New York.
FUORTES, E.; University of Naples.
GALTON, FRANCIS, M.A., F.R.G.S.; London.
GARDINER, MARTIN; late Professor of Mathematics in St. John's College, Sydney.
GENESE, R. W., B.A.; Scholar of St. John's Coll., Cambridge.
GLAISHER, J. W. L., B.A.; Fellow of Trinity College, Cambridge.
GLASHAN, J. C.; Strathroy, Ontario.
GODFRAY, HUGH, M.A.; Cambridge.
GODWARD, WILLIAM; Chelsea.
GRANT, S. C. W.; Royal Naval School, New Cross.
GREAVES, MYRA E.; Rue de la Paix, Boulogne.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITH, W.; Superintendent of Public Schools, New London, Ohio, United States.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
HALL, ASAPH, M.A.; Observatory, Washington.
HAMMOND, J., M.A.; Buckhurst Hill, Essex.
HANNA, W.; Literary Institute, Belfast.
HARKEMA, C.; University of St. Petersburgh.
HARLEY, Rev. ROBERT, F.R.S.; Vice-Master of Mill Hill Grammar School.
HARRIS, H. W.; Trinity College, Dublin.
HART, Dr. DAVID S.; Stonington, Connecticut.
HART, H.; R.M. Academy, Woolwich.
HERMITE, Ch.; Membre de l'Institut, Paris.
HILARY, H., M.A.; Dry Hill Park, Tonbridge.
HILL, Rev. E., M.A.; St. John's College, Camb.
HINTON, C. H.; Savile Row, London.
HIRST, Dr. T. A., F.R.S.; Director of Studies in the Royal Naval College, Greenwich.
HOGG, W., M.A.; Royal Naval School, New Cross.
HOPKINS, Rev. G. H., M.A.; Egham.
HOPKINSON, J., D.Sc., B.A.
HOSKINS, HENRY; Notting Hill Square, London.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Fellow of St. John's College, Cambridge.
INGLEBY, C. M., M.A., LL.D.
INGLEBY, O. E.; Jersey.
JAMESON, JOHN; Essex Bridge Works, Dublin.
JENKINS, MORGAN, M.A.; late Scholar of Christ's Coll. Camb., Westbourne Park, London.
JENKINS, J. S., L.C.P.; Malvern.
JOHNSON, J. M., B.A.; Radley College, Abingdon.

Johnstone, E.; Chaplain, Southsea.
Jones, Llewellyn W.; Ayr.
Kelland, Philip, M.A.; Professor of Mathematics in the University of Edinburgh.
King, Q. W.; Royal Hospital School, Greenwich.
Kirkman, Rev. T. P., M.A., F.R.S.; Croft Rectory, near Warrington.
Kitchin, Rev. J. L., M.A.; Head Master of Bideford Grammar School.
Kittudge, Lizzie A.; Boston, United States.
Knisely, Rev. U. J.; Newcomerstown, Ohio.
Ladd, Christine; Massachusetts.
Laverty, W. H., M.A.; Public Examiner in the University of Oxford.
Leudesdorf, C., M.A.; Fellow of Pembroke College, Oxford.
McCay, W. S., B.A.; Trinity College, Dublin.
McColl, Hugh; Rue du Temple, Boulogne.
McCormick, E.; Ledbury, Hereford.
McDowell, J., M.A.; Pembroke Coll., Camb.
McLean, Abijah; New Lisbon, Ohio.
McLeod, J., M.A.; R.M. Academy, Woolwich.
Mackenzie, J. L.; Independent Coll., Taunton.
Madden, W. M.; Trinity Parsonage, Wakefield.
Malet, J. C., M.A.; Trinity College, Dublin.
Mannheim, M.; Professeur à l'Ecole Polytechnique, Paris.
Martin, Artemas; "Editor of Higher Mathematics" in *Normal Monthly*, Erie.
Martin, Rev. H., D.D., M.A.; Examiner in Mathematics in the University of Edinburgh.
Mason, J.; East Castle Colliery, Newcastle.
Mathews, F. C., M.A.; London.
Matteson, Dr. James; De Kalb Centre, Illinois.
Merrifield, C. W., F.R.S.; Kensington, London.
Merrifield, J., LL.D., F.R.A.S.; Plymouth.
Merrick, Thos.; Kensington Square, London.
Miller, W. J. C., B.A.; Parkfield, Westhill, Huddersfield.
Minchin, G.M., M.A.; Prof. in Cooper's Hill Coll.
Moon, Robert, M.A.; late Fellow of Queen's College, Cambridge, Cleveland Sq., London.
Monck, Henry Stanley; Trinity Coll., Dublin.
Morley, Thos., L.C.P.; Bromley, Kent.
Moulton, J. F., M.A.; Fellow of Christ's Coll., Cambridge.
Mulcaster, J.; Woolwich Common.
Murphy, Hugh; Head Master of the Incorporated Society's School, Dublin.
Nash, A. M., B.A.; Professor of Nat. Phil. and Astronomy, Presidency College, Calcutta.
Nelson, R. J., M.A.; Naval School, London.
O'Cavanagh, Patrick; Dublin.
Ogilvie, G. A.; Leiston, Saxmundham.
Otter, W. Curtis, F.R.A.S.; London.
Panton, A. W., M.A.; Trinity College, Dublin.
Phillips, F. B. W.; Balliol College, Oxford.
Pirie, A., M.A.; Univ. of St. Andrews.
Polignac, Prince Camille de; Paris.
Prince, J.; Andover, Hants.
Prudden, Frances E.; Lockport, New York.
Renshaw, S. A.; Nottingham.
Rhodes, Ellen; Helensborough.
Rippin, Charles R., M.A.; Woolwich Common.
Roberts, Samuel, M.A.; Holloway, London.
Roberts, Rev. W., M.A.; Fellow and Senior Tutor, Trinity College, Dublin.
Roberts, W., Junior; Trinity College, Dublin.
Roberts, R. A.; Trin. Coll., Dublin.
Rosenthal, L.H.; Scholar of Trin. Coll., Dublin.
Rücker, A. W., B.A.; Professor of Mathematics in the Yorkshire College of Science, Leeds.
Rutter, Edward; Sunderland.
Salmon, Rev. G., D.D., F.R.S.; Regius Professor of Divinity in the University of Dublin.
Sanders, J. B.; Bloomington, Indiana.
Sanderson, George; Gallashiels.
Sanderson, Rev. T. J., M.A.; Royston, Cambs.
Sarkar, Nilkantha, B.A.; Presidency College, Calcutta.
Savage, Thomas, M.A.; Fellow of Pembroke College, Cambridge.
Scott, Josiah; Judge of the Ohio Supreme Court, Bucyrus, United States.
Scott, R. F., B.A.; St. John's College, Camb.
Seitz, E. B.; Greenville, Ohio, United States.
Serret, Paul; Paris.
Shepherd, A. J. P.; Queen's College, Oxford.
Sides, J. J.; Rue des Vieillards, Boulogne.
Siverly, Walter; Oil City, Pennsylvania.
Slade, J. H.; Royal Hospital School, Greenwich.
Spottiswoode, William, M.A., F.R.S.; Grosvenor Place, London.
Stein, A.; Venice.
Symes, R. W., B.A.; London.
Sylvester, J. J., LL.D., F.R.S.; Member of the Institute of France, &c.
Tait, P. G., M.A.; Professor of Natural Philosophy in the University of Edinburgh.
Tarleton, Francis A., M.A.; Fellow of Trinity College, Dublin.
Taylor, Rev. C., M.A.; Fellow of St. John's College, Cambridge.
Taylor, H. M., M.A.; Fellow and Assistant Tutor of Trinity College, Cambridge.
Taylor, J. H., B.A.; Cambridge.
Tebay, Septimus, B.A.; Head Master of Rivington Grammar School.
Thomson, F. D., M.A.; late Fellow of St. John's Coll., Camb., Brinkley Rectory, Newmarket.
Timothy, J. E.; Lockport, New York.
Todhunter, Isaac, F.R.S.; Cambridge.
Tomlinson, H.; Christ Church College, Oxford.
Torelli, Gabriel; University of Naples.
Torry, Rev. A. F., M.A.; St. John's Coll., Camb.
Townsend, Rev. R., M.A., F.R.S.; Professor of Nat. Phil. in the University of Dublin.
Traill, Anthony, M.A., M.D.; Fellow of Trinity College; Dublin.
Trowbridge, David; Waterburgh, New York.
Tucker, R., M.A.; Mathematical Master in University College School, London.
Turrell, I. H.; Cumminsville, Ohio.
Vose, G. B.; Professor of Mechanics and Civil Engineering, Washington, United States.
Walker, J. J., M.A.; Vice-Principal of Univ. Hall.
Walmsley, J., B.A.; Eccles, Manchester.
Ward, Isabella M.; Capecure, Boulogne.
Warren, R., M.A.; Trinity College, Dublin.
Watson, Stephen; Haydonbridge.
Watson, Rev. H. W., M.A.; late Fellow of Trinity College, Cambridge.
Wertsch, Fr.; Weimar.
White, Rev. J., M.A.; Woolwich.
Whitworth, Rev. W. A., M.A.; Fellow of St. John's College, Cambridge.
Wilkinson, Rev. M. M. U.; Norwich.
Williams, S. F.; Liverpool College.
Williamson, B., M.A.; Fellow of Trinity College, Dublin.
Wilson, J. M., M.A.; Rugby School.
Wilson, Rev. J., M.A.; Rector of Bannockburn Academy.
Wilson, Rev. J. R., M.A.; Royston, Cambs.
Wilson, Rev. R., D.D.; Chelsea.
Wolstenholme, Rev. J., M.A.; Professor of Mathematics in Cooper's Hill College.
Woolhouse, W. S. B., F.R.A.S., &c.; London.
Wordingham, Elizabeth; Fakenham.
Worth, J. E.; Borough Engineers' Office, Nottingham.
Wright, Dr. S. H., M.A.; Penn Yan, New York.
Wright, W. E.; Royal School, Raphoe.

CONTENTS.

Mathematical Papers, &c.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

4666. (By Prof. TOWNSEND.)—An elastic uniform beam MN has its extremities M and N and any intermediate point O kept in the same horizontal right line by three rigid supports; shew that its two points of inflexion Y and Z, in its position of strained equilibrium under the action of gravity, are the two inverses of its extremities M and N with respect to the circle round O as centre the square of whose diameter $= \dfrac{m^3+n^3}{m+n}$, where m and n are the lengths of its two segments OM and ON determined by the intermediate point O.

Solution by the PROPOSER.

Taking the horizontal line MON and the vertical through O as axes of coordinates, and denoting by l the length of the beam, by W its weight, by E its modulus of elasticity, by I the moment of inertia of its tranverse area round the horizontal axis passing through its centre of inertia, and by P and Q the reactions of the supports at M and N; we have, for the differential equations of equilibrium of the two segments OM and ON, respectively,

$$\mathrm{EI}\,.\,\frac{d^2y}{dx^2} = \mathrm{P}\,(m-x) - \tfrac{1}{2}\,\frac{\mathrm{W}}{l}\,(m-x)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

$$\mathrm{EI}\,.\,\frac{d^2z}{dx^2} = \mathrm{Q}\,(n+x) - \tfrac{1}{2}\,\frac{\mathrm{W}}{l}\,(n+x)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

from which, if P and Q were known, the positions of Y and Z on OM and ON would be at once determined by equating to 0 their right-hand members, and solving for x in each case.

Since at the origin their left-hand members are evidently equal, therefore

$$\mathrm{P}m - \mathrm{Q}n = \tfrac{1}{2}\,\frac{\mathrm{W}}{l}\,(m^2-n^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3),$$

which is one relation by which to determine the values of P and Q.

To find another relation between them, integrating once (1) and (2)

with respect to x, we have

$$\mathrm{EI}\,.\,\frac{dy}{dx} = -\tfrac{1}{2}\,\mathrm{P}\,.\,(m-x)^2 + \tfrac{1}{6}\,\frac{\mathrm{W}}{l}\,(m-x)^3 + \mathrm{C} \quad\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$\mathrm{EI}\,.\,\frac{dz}{dx} = +\tfrac{1}{2}\,\mathrm{Q}\,.\,(n+x)^2 - \tfrac{1}{6}\,\frac{\mathrm{W}}{l}\,(n+x)^3 - \mathrm{D} \quad\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

the two constants of integration C and D, since at the origin the two left-hand members are again evidently equal, being connected by the relation

$$\mathrm{C}+\mathrm{D} = \tfrac{1}{2}\,(\mathrm{P}m^2+\mathrm{Q}n^2) - \tfrac{1}{6}\,\frac{\mathrm{W}}{l}\,(m^3+n^3) \quad\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and integrating again (4) and (5) with respect to x, we have

$$\mathrm{EI}\,.\,y = \tfrac{1}{6}\,\mathrm{P}\,(m-x)^3 - \tfrac{1}{24}\,\frac{\mathrm{W}}{l}\,(m-x)^4 - \mathrm{C}\,(m-x) \quad\ldots\ldots\ldots\ldots\ldots (7),$$

$$\mathrm{EI}\,.\,z = \tfrac{1}{6}\,\mathrm{Q}\,(n+x)^3 - \tfrac{1}{24}\,\frac{\mathrm{W}}{l}\,(n+x)^4 - \mathrm{D}\,(n+x) \quad\ldots\ldots\ldots\ldots\ldots (8).$$

Since $y=0$ when $x=m$, and $z=0$ when $x=-n$; but since also $y=0$ and $z=0$ when $x=0$, therefore

$$\mathrm{C} = \tfrac{1}{6}\,\mathrm{P}m^2 - \tfrac{1}{24}\,\frac{\mathrm{W}}{l}\,m^3, \quad\text{and}\quad \mathrm{D} = \tfrac{1}{6}\,\mathrm{Q}n^2 - \tfrac{1}{24}\,\frac{\mathrm{W}}{l}\,n^3 \quad\ldots\ldots\ldots\ldots (9),$$

adding which, and equating C + D to its value above found, we get

$$\mathrm{P}m^2 + \mathrm{Q}n^2 = \tfrac{3}{8}\,\frac{\mathrm{W}}{l}\,(m^3+n^3) \quad\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

which is the other relation by which to determine the values of P and Q.

Solving for P and Q from (3) and (10), we get

$$\mathrm{P} = \tfrac{1}{8}\,\mathrm{W}\,.\,\frac{3m^2+mn-n^2}{m\,(m+n)}, \quad \mathrm{Q} = \tfrac{1}{8}\,\mathrm{W}\,.\,\frac{3n^2+nm-m^2}{n\,(n+m)} \quad\ldots\ldots\ldots\ldots\ldots (11),$$

which, substituted in the right-hand members of (1) and (2) equated to 0, give for the distances OY and OZ from O of Y and Z on OM and ON respectively, the values

$$\mathrm{OY} = \frac{1}{4m}\,.\,\frac{m^3+n^3}{m+n}, \quad \mathrm{OZ} = -\,\frac{1}{4n}\,.\,\frac{m^3+n^3}{m+n} \quad\ldots\ldots\ldots\ldots (12),$$

from which, since OM $= m$ and ON $= -n$, it follows that

$$\mathrm{OM}\,.\,\mathrm{OY} = \mathrm{ON}\,.\,\mathrm{OZ} = \tfrac{1}{4}\,.\,\frac{m^3+n^3}{m+n} \quad\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

which of course establishes the property in question.

In the particular case when $m=n$, that is, when the intermediate point of support O is midway between the extremities M and N of the beam; then, as appears at once from (12) above, OY $= \tfrac{1}{4}$OM and OZ $= \tfrac{1}{4}$ON; which are their well known values in that comparatively simple case.

If R be the reaction of the support at the intermediate point O, whatever be its position on the beam, since always P + Q + R = W, it follows at once from (11) that

$$\mathrm{R} = \tfrac{1}{8}\,\mathrm{W}\,.\,\frac{m^2+3mn+n^2}{mn} \quad\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

the same value obtained on other principles in the answer to Question 4067 (see *Reprint*, Vol. XXI., p. 78).

In the particular case when $m=n$, then, as appears at once from (11) and (14) above, $\mathrm{P}=\mathrm{Q}=\tfrac{3}{16}\mathrm{W}$, and $\mathrm{R}=\tfrac{10}{16}\mathrm{W}$; which, as is well known, are their values in that case.

4697. (By Prof. Wolstenholme.)—Prove that

$$\int_1^{\sqrt{2}} \frac{(x^4-2x^2+2)^n}{x^{2n+1}}\,dx = \int_1^{\sqrt{2}} \frac{(x^2-2x+2)^n}{x^{n+1}}\,dx.$$

Solution by E. B. Elliott.

This is included in the following theorem:—If $S(a, b)$ be any symmetric function of a and b, then we have

$$\int_1^n S\left(x^2, \frac{n^2}{x^2}\right)\frac{dx}{x} = \int_1^n S\left(x, \frac{n^2}{x}\right)\frac{dx}{x}.$$

The transformations $x^2 = y$, and then $\frac{n^2}{x} = y$, give

$$\int_1^n S\left(x^2, \frac{n^2}{x^2}\right)\frac{dx}{x} = \frac{1}{2}\int_1^{n^2} S\left(y, \frac{n^2}{y}\right)\frac{dy}{y},$$

$$\int_1^n S\left(x, \frac{n^2}{x}\right)\frac{dx}{x} = -\int_{n^2}^n S\left(\frac{n^2}{y}, y\right)\frac{dy}{y} = \int_n^{n^2} S\left(y, \frac{n^2}{y}\right)\frac{dy}{y}$$

$$= \frac{1}{2}\left\{\int_1^n S\left(x, \frac{n^2}{x}\right)\frac{dx}{x} + \int_n^{n^2} S\left(y, \frac{n^2}{y}\right)\frac{dy}{y}\right\}$$

$$= \frac{1}{2}\int_1^{n^2} S\left(y, \frac{n^2}{y}\right)\frac{dy}{y}.$$

Hence
$$\int_1^n S\left(x^2, \frac{n^2}{x^2}\right)\frac{dx}{x} = \int_1^n S\left(x, \frac{n^2}{x}\right)\frac{dx}{x}.$$

In particular, $\displaystyle\int_1^{\sqrt{2}} \left(x^2 - 2 + \frac{2}{x^2}\right)^n \frac{dx}{x} = \int_1^{\sqrt{2}} \left(x - 2 + \frac{2}{x}\right)^n \frac{dx}{x}$,
which proves the theorem in the question.

4678. (By S. Tebay.)—A clepsydra is regulated to run 24 hours; if the tap in the bottom be turned on at random, shew that the probable time indicated by the instrument in the time t is

$$24\left[1 - \left(\frac{48}{5t}\right)^{\frac{2}{3}}\left\{1 - \left(1 - \frac{t}{24}\right)^{\frac{5}{2}}\right\}^{\frac{2}{3}}\right].$$

Solution by the Proposer.

Let h be the length of the index scale, $h - x$ the depth of the water, and y the radius of the surface at the time t; $v = \frac{dx}{dt}$ when the area of the

orifice is a; A, V the greatest values of a, v. Then the equation to the generating curve of the clepsydra is

$$\pi y^2 = \frac{A}{V}\left\{2g(h-x)\right\}^{\frac{1}{2}}.$$

It can be easily shewn that $\frac{v}{V} = \frac{a}{A}$ when a is invariable, so that we can write $x = vt$. Hence the quantity of water discharged through the orifice a in the time t is

$$\int a\left\{2g(h-vt)\right\}^{\frac{1}{2}} dt = \frac{2}{3}\,\frac{a}{v}\,(2g)^{\frac{1}{2}}\left\{h^{\frac{3}{2}} - (h-vt)^{\frac{3}{2}}\right\}$$

$$= \frac{2}{3}\,\frac{A}{V}\,(2g)^{\frac{1}{2}}\left\{h^{\frac{3}{2}} - (h-vt)^{\frac{3}{2}}\right\}.$$

Multiply by $\frac{dv}{V}$, and integrate from $v = 0$, to $v = V = \frac{h}{24}$; then the average discharge in the time t is

$$16A(2gh)^{\frac{1}{2}}\left[1 - \frac{48}{5t}\left\{1 - \left(1 - \frac{t}{24}\right)^{\frac{5}{2}}\right\}\right].$$

Let T be the time in which the average is discharged when the tap is turned on at full; then

$$16A(2gh)^{\frac{1}{2}}\left\{1 - \left(1 - \frac{T}{24}\right)^{\frac{3}{2}}\right\} = 16A(2gh)^{\frac{1}{2}}\left[1 - \frac{48}{5t}\left\{1 - \left(1 - \frac{t}{24}\right)^{\frac{5}{2}}\right\}\right];$$

and therefore $\quad T = 24\left[1 - \left(\frac{48}{5t}\right)^{\frac{2}{3}}\left\{1 - \left(1 - \frac{t}{24}\right)^{\frac{5}{2}}\right\}^{\frac{2}{3}}\right].$

4649. (By J. J. Walker.)—Prove that, if α, β, γ are any three plane angles,

$$\cos 2\alpha \cos^2(\beta+\gamma) + \cos 2\beta \cos^2(\gamma+\alpha) + \cos 2\gamma \cos^2(\alpha+\beta)$$
$$- \cos 2\alpha \cos 2\beta \cos 2\gamma = 2\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta).$$

Solution by R. F. Davis; R. Tucker; R. W. Genese; *and others.*

$$2\,(\text{left hand member}) = \cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2(\alpha+\beta+\gamma)$$
$$= 4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha).$$

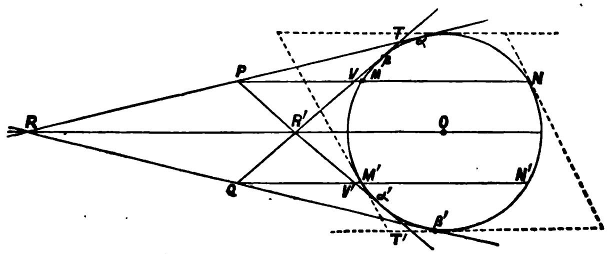
4476. (Proposed by Alpha.)—One focal chord of a conic meets the tangents at the extremities of another in A, B; if straight lines ACD, BEF be drawn perpendicular to AB, meeting the curve in C, D, E, F; then prove (1) that CE and DF meet AB on the directrix, in a point P; (2) that CF and DE, AF and DB, AE and BC meet on the polar of P; (3) that the intercepts CD, EF subtend equal angles at the focus S; (4) that SA : SC : SD = SB : SE : SF; (5) that CF and DE meet AB in two

points G, H, having properties like those of A, B; and (6) that of the four intersections of tangents drawn through A, B, two will lie on the polar of P, and two on the directrix.

Solution by C. LEUDESDORF; S. FORDE; *and others.*

Reciprocating the conic into a circle, we have this theorem:—

Two pairs of parallel tangents are drawn to a circle; through M′ and N, the points of contact of one pair, straight lines M′N′, MN are drawn



parallel to the direction of the other pair. From S, any fixed point, the perpendicular SPQ is drawn to M′N′, MN; and from P and Q are drawn Pα, Pα', Qβ, Qβ' to the circle. Completing the figure, we see that,

1. The straight line RR′ passes through O.
2. TT′, VV′, WW′ meet in the pole of OR′R (W, W′ being the intersections of MN, QT′; M′N′, PT.
3. The angles αPα', βQβ', are equal.
4. The ratio of the perpendiculars from S on Pα or Pα' to those on Qβ' or Qβ is equal to SP : SQ.
5. The straight lines drawn through T and T′ parallel to MN have similar properties to those possessed by MN, M′N′.
6. Of the four straight lines MM′, MN′, NN′, NM′, two meet in O, and two in the pole of OR′R.

4632. (By J. L. McKENZIE.)—If r and r' are the focal radii to the extremities of two conjugate semi-diameters of an ellipse, and ρ and ρ' the radii to the corresponding points on the nth focal pedal, prove that

$$\left(\frac{r}{\rho}\right)^{\frac{2}{n}} + \left(\frac{r'}{\rho'}\right)^{\frac{2}{n}} = \frac{a^2+b^2}{b^2}.$$

Solution by R. F. DAVIS; *the* PROPOSER; *and others.*

Let r be the radius vector of any point of the ellipse, and p the perpendicular from the focus upon the tangent at that point. Then if r_m, p_m be corresponding quantities for the corresponding point on the mth focal

pedal, we have $\frac{p_n}{r_n} = \frac{p_{n-1}}{r_{n-1}} = \ldots\ldots = \frac{p_1}{r_1} = \frac{p}{r}$, and $r_i = p_{i-1}$.

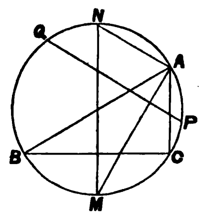
Hence $\left(\frac{r}{r_n}\right)^{\frac{1}{n}} = \frac{r}{p} = \operatorname{cosec}\phi$, where ϕ is the angle between the radius vector and the tangent. Thus

$$\left(\frac{r}{r_n}\right)^{\frac{2}{n}} + \left(\frac{r'}{r'_n}\right)^{\frac{2}{n}} = \operatorname{cosec}^2\phi + \operatorname{cosec}^2\phi' = \frac{\delta'^2}{b^2} + \frac{\delta^2}{b^2}, = \frac{a^2+b^2}{b^2};$$

where δ, δ' are the lengths of the semi-diameters through the two points.

4051. (Proposed by J. McDowell.)—If on a circle drawn round the triangle ABC, P and Q be two points such that the distance of P or Q from A is a mean proportional between its distances from B and C, prove that the difference between the angles PAB, QAC is half the difference between ABC, ACB.

Solution by R. W. Genese; R. F. Davis; *and others.*



Let M be the middle point of the arc BC; then $PB.PC = PM^2 - MC^2$, therefore $MC^2 = PM^2 - PA^2 = QM^2 - QA^2$. We see then that P and Q lie on a perpendicular to AM; and the property in the Question is true if PQ be any chord perpendicular to AM. Draw AN parallel to PQ; then, considering arcs $PB \sim QC = QM \sim PM = MN - MA = AN = \frac{1}{2}(AB \sim AC)$; therefore, &c.

4652. (By W. H. H. Hudson.)—From the vertex O and any point P on a catenary, ordinates OA, PN are drawn to the directrix; prove that the area OANP = OA . OP.

Solution by R. W. Genese; R. Tucker; *and others.*

Let P_1P_2 be an infinitesimal arc, P_1N_1, P_2N_2 the corresponding ordinates. From N_1 draw N_1T perpendicular to P_1P_2; then, to the first order of small quantities, we have

$$\text{area } P_1N_1N_2P_2 = \text{rect. } N_1T \, . \, P_1P_2.$$

But P_1P_2 is ultimately (when P_2 is made to approach P_1) the tangent at P_1. And, by a known property of the catenary, N_1T is thus $= OA$, therefore area $P_1N_1N_2P_2 = \text{rect. } OA \, . \, P_1P_2$. Summing all such areas, we get the result in question.

4524. (By R. TUCKER.)—A parabola and its vertical inverse ($rr' = k^2 = 8a^2$) are taken; prove that, if ρ, ρ' be the radii of curvature corresponding to the point where the inverse meets the latus rectum of the parabola, then we shall have $\rho = 6\rho'$.

Solution by the PROPOSER; Rev. T. J. SANDERSON; *and others.*

In art. 38 of my article on Radials (*Reprint*, Vol. II., p. 30), it is shown that, for the curve in the question, $\rho \tan^4 \theta = 6\rho'$; but the inverse curve cuts the latus rectum of the parabola when $\theta = \frac{1}{4}\pi$; hence, under the given condition, $\rho = 6\rho'$.

4627. (By the Rev. T. J. SANDERSON.)—In the Cambridge Mathematical Tripos List, the names in each class are arranged in order of merit. What is the chance (1) that each class is also in alphabetical order, (2) that the whole list is in alphabetical order, and (3) that in an alphabetical list of the whole, every man is in order of merit with respect to the men of his own class?

Solution by the PROPOSER.

1. Let p, q, r be the numbers in each class; then, by permuting each class, the number of arrangements possible is $\underline{|p} \times \underline{|q} \times \underline{|r}$; therefore the chance required $= \dfrac{1}{\underline{|p}\,\underline{|q}\,\underline{|r}}$.

2. By permuting the whole list we get $\underline{|p+q+r}$ different arrangements; therefore the chance required $= \dfrac{1}{\underline{|p+q+r}}$.

3. There being $p+q+r$ places in the list, the 1st class can be arranged (in order of merit always) in $\dfrac{\underline{|p+q+r}}{\underline{|p}\,\underline{|q+r}}$ ways. There being then $q+r$ places left, the 2nd class can be arranged in $\dfrac{\underline{|q+r}}{\underline{|q}\,\underline{|r}}$ ways. The 3rd class can then be arranged in the remaining r places in 1 way. And each of the above arrangements may occur with any one of the others; therefore the whole chance required $= \dfrac{\underline{|p+q+r}}{\underline{|p}\,\underline{|q}\,\underline{|r}} \times \dfrac{1}{\underline{|p+q+r}} = \dfrac{1}{\underline{|p}\,\underline{|q}\,\underline{|r}}$.

4488. (By M. JENKINS, M.A.)—Using the theorems that the centre of gravity of the solid content of a uniform tetrahedron coincides with the

centre of gravity of particles of equal weight placed at the corners, prove (1) that if A, a be the areas of the ends, M the area of the parallel mid-section of any frustum of a uniform pyramid or cone, H, h the centres of gravity of A, a respectively, then G, the centre of gravity of the frustum, lies in the line Hh, and divides it so that $\dfrac{\text{HG}}{h\text{G}} = \dfrac{a+2\text{M}}{\text{A}+2\text{M}}$. Also prove (2) the following rule for finding the centre of gravity of any uniform wedge, that is, the solid figure obtained by joining towards the same parts any three parallel straight lines not all lying in the same plane: "Let P, Q, R be the mid-points of the three parallel edges of lengths l, m, and n respectively, g the centre of gravity of the triangle PQR and therefore also of particles of equal weight placed at the six corners of the wedge, h the centre of gravity of weights l, m, and n, placed at P, Q, and R respectively, and therefore also of uniform rods coinciding with the three parallel edges of the wedge, then G the centre of gravity of the wedge lies in gh and divides it so that $g\text{G} = \frac{1}{4}gh$."

Solution by the Proposer.

1. Let PQR, pqr be the similar triangular faces of the frustum of a triangular pyramid; join Pq, Pr, qR; then the frustum is divided into three triangular pyramids Ppqr, PRqr, PQqR, and $\dfrac{\text{pyramid } p\text{P}qr}{\text{pyramid PR}qr} = \dfrac{rp}{\text{PR}} = \dfrac{a^{\frac{1}{2}}}{\text{A}^{\frac{1}{2}}}$; because the pyramids have a common base Pqr, and pr, RP are parallel lines drawn from their vertices to their base: in like manner

$$\frac{\text{pyramid } r\text{P}q\text{R}}{\text{pyramid QP}q\text{R}} = \frac{qr}{\text{QR}} = \frac{a^{\frac{1}{2}}}{\text{A}^{\frac{1}{2}}};$$

therefore $$\frac{\text{pyramid } p\text{P}qr}{a} = \frac{\text{pyramid PR}qr}{\text{A}^{\frac{1}{2}}a^{\frac{1}{2}}} = \frac{\text{pyramid QP}q\text{R}}{\text{A}}.$$

Hence G coincides with the centre of gravity of weights A placed at Q, P, q, and R; $\text{A}^{\frac{1}{2}}a^{\frac{1}{2}}$ at P, R, q, and r; and a at p, P, q, and r.

In like manner joining qr to P, but joining Qr instead of qR, we shall obtain another system of weights; and in like manner two more systems by joining pq to R, and two more by joining pr to Q, according to the following table:—

At P	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	A	$\text{A}+(\text{A}a)^{\frac{1}{2}}$	$\text{A}+(\text{A}a)^{\frac{1}{2}}$	A
„ Q	A	$\text{A}+(\text{A}a)^{\frac{1}{2}}$	$\text{A}+(\text{A}a)^{\frac{1}{2}}$	A	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$
„ R	$\text{A}+(\text{A}a)^{\frac{1}{2}}$	A	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	A	$\text{A}+(\text{A}a)^{\frac{1}{2}}$
„ p	a	a	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	$(\text{A}a)^{\frac{1}{2}}+a$	$(\text{A}a)^{\frac{1}{2}}+a$	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$
„ q	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	$(\text{A}a)^{\frac{1}{2}}+a$	$(\text{A}a)^{\frac{1}{2}}+a$	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	a	a
„ r	$(\text{A}a)^{\frac{1}{2}}+a$	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	a	a	$\text{A}+(\text{A}a)^{\frac{1}{2}}+a$	$(\text{A}a)^{\frac{1}{2}}+a$

Combining these weights, and cutting out the factor 2, G coincides with the centre of gravity of weights $3\text{A}+2(\text{A}a)^{\frac{1}{2}}+a$ at P, Q and R, together with $\text{A}+2(\text{A}a)^{\frac{1}{2}}+3\text{A}$ at p, q and r.

But $m^{\frac{1}{2}} = \frac{1}{2}(\text{A}^{\frac{1}{2}}+a^{\frac{1}{2}})$, therefore $3\text{A}+2(\text{A}a)^{\frac{1}{2}}+a = 2\text{A}+4\text{M}$, and $\text{A}+2(\text{A}a)^{\frac{1}{2}}+3a = 2a+4\text{M}$, whence follows the required construction for the frustum of any triangular pyramid.

The theorem will then hold for the frustum of any polygonal pyramid or of any cone, since, if we compared the given frustum with the frustum of a triangular pyramid obtained by joining two similar and similarly situated triangles described about the boundaries of the ends of the given frustum, the sections of the given frustum would bear a constant ratio to the sections of the frustum of the triangular pyramid. It is interesting to notice the exact analogy which holds between the formula here proved and the formula for the centre of gravity of a quadrilateral with two sides parallel.

2. Let $AB = l$, $CD = m$, and $EF = n$ be the three parallel edges, A, C, and E being towards the same parts, and B, D, and F towards the same parts. Then, if the wedge be divided into three triangular pyramids AFDB, AFDC, and AFCE, having a common edge AF, then the first two pyramids having a common base AFD have their altitudes proportional to BA and CD drawn in parallel directions from their vertices B and C to their bases: in like manner the last two pyramids have a common base and their altitudes proportional to CD and EF; hence the solid contents of the pyramids are proportional to l, m, and n. Thus G will coincide with the centre of gravity of weights l at A, F, D, and B; m at A, F, D, and C; and n at A, F, C, and E. In like manner we can divide the wedge into three triangular pyramids in five other ways according to the edge which is common to the three triangular pyramids, and the weights will be arranged according to the following table:—

Com. Edge	AF	BE	AD	BC	CF	DE
At A	$l+m+n$	l	$l+m+n$	l	$l+n$	$l+m$
„ B	l	$l+m+n$	l	$l+m+n$	$l+m$	$l+n$
„ C	$m+n$	$l+m$	m	$l+m+n$	$l+m+n$	m
„ D	$l+m$	$m+n$	$l+m+n$	m	m	$l+m+n$
„ E	n	$l+m+n$	$m+n$	$l+n$	n	$l+m+n$
„ F	$l+m+n$	n	$l+n$	$m+n$	$l+m+n$	n

Combining these weights and cutting out the common factor 3, G will coincide with the centre of gravity of weights

$l+m+n+l$ at A and B; $l+m+n+m$ at C and D; $l+m+n+n$ at E and F; and therefore, with that of $l+m+n$ at P, Q, and R, together with l at P, m at Q, and n at R, and therefore with that of $3(l+m+n)$ at g, together with $l+m+n$ at h, whence the required construction follows.

4642. (By Prof. Townsend.)—A uniform horizontal beam is supported at its centre and both ends by three uniform and equal vertical props; find the distribution of its weight between them, taking into account its elasticity and theirs.

Solution by the Proposer.

Denoting by a the semi-length, by α the transverse area, and by μ the modulus of elasticity of the beam; by b the common length, by β the transverse area, and by ν the modulus of elasticity of the props; by P, Q, R

the required components of the weight W of the beam at its two extremities A and B and at its centre C respectively; by p, q, r the three corresponding contractions of the three props arising from their common elasticity; and by k the radius of flexural inertia of α round its centre of gravity; then, since, on elementary principles of longitudinal elasticity,

$$p = \mathrm{P}\cdot\frac{b}{\beta\nu},\quad q = \mathrm{Q}\cdot\frac{b}{\beta\nu},\quad r = \mathrm{R}\cdot\frac{b}{\beta\nu}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

and since, evidently, $\mathrm{P} = \mathrm{Q}$, and $\mathrm{P}+\mathrm{Q}+\mathrm{R} = \mathrm{W}$; therefore

$$(r-p) = (\mathrm{R}-\mathrm{P})\cdot\frac{b}{\beta\nu} = \tfrac{1}{2}(3\mathrm{R}-\mathrm{W})\cdot\frac{b}{\beta\nu}\ldots\ldots\ldots\ldots\ldots\ldots\ (2);$$

but since, on known principles of transverse deflection, $(r-p)$ cannot exceed positively $\frac{5}{48}\cdot\frac{a^3}{k^2a\mu}\cdot\mathrm{W}$, which would be its value if the central prop were removed, in which case $\mathrm{P} = \mathrm{Q} = \frac{1}{2}\mathrm{W}$, and $\mathrm{R} = 0$, or negatively $\frac{3}{48}\cdot\frac{a^3}{k^2a\mu}\cdot\mathrm{W}$, which would be its value if the two extreme props were removed, in which case $\mathrm{P} = \mathrm{Q} = 0$, and $\mathrm{R} = \mathrm{W}$; therefore for any intermediate value of $(r-p)$ the corresponding value of R is given by the proportion

$$\mathrm{R} : \mathrm{W} = \left[\frac{5}{48}\cdot\frac{a^3}{k^2a\mu}\cdot\mathrm{W} - (r-p)\right] : \frac{8}{48}\cdot\frac{a^3}{k^2a\mu}\cdot\mathrm{W}\ \ldots\ldots\ (3);$$

from which, substituting for $(r-p)$ its value given by (2), and solving for R, we get $\left[\frac{1}{3}\,\frac{a^3}{k^2a\mu} + 3\,\frac{b}{\beta\nu}\right]\mathrm{R} = \left[\frac{10}{48}\,\frac{a^3}{k^2a\mu} + \frac{b}{\beta\nu}\right]\mathrm{W}\ldots\ldots\ldots\ldots\ (4)$;

and, since $\mathrm{P} = \mathrm{Q} = \frac{1}{2}(\mathrm{W}-\mathrm{R})$, consequently

$$\left[\frac{1}{3}\,\frac{a^3}{k^2a\mu} + 3\,\frac{b}{\beta\nu}\right]\mathrm{P} = \left[\frac{3}{48}\,\frac{a^3}{k^2a\mu} + \frac{b}{\beta\nu}\right]\mathrm{W}\ldots\ldots\ldots\ldots\ (5),$$

which, substituting Q for P in the latter, give accordingly the distribution required.

In the extreme case when $\mu = \infty$, that is, when the beam is perfectly rigid, then $\mathrm{P} = \mathrm{Q} = \mathrm{R} = \frac{1}{3}\mathrm{W}$, their known values in that case.

In the other extreme case, when $\nu = \infty$, that is, when the props are perfectly rigid, then $\mathrm{R} = \frac{10}{16}\mathrm{W}$, and $\mathrm{P} = \mathrm{Q} = \frac{3}{16}\mathrm{W}$, their known values in that case also.

3618. (By S. Watson.)—Find the locus and area of the intersection of a normal to an ellipse and a perpendicular upon it, (1) from the centre, (2) from the focus.

Solution by the Proposer.

1. Let (x', y') be the point where the normal is drawn; then the equations of the normal and perpendicular upon it are

$$y - y' = \frac{a^2y'}{b^2x'}(x-x'),\ \text{and}\ y = -\frac{b^2x'}{a^2y'}\,x\ldots\ldots\ldots\ldots\ (1, 2).$$

From the last of these and $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$, we get

$$x' = \pm \frac{a^2y}{(a^2y^2+b^2x^2)^{\frac{1}{2}}},\quad y' = \mp \frac{b^2x}{(a^2y^2+b^2x^2)^{\frac{1}{2}}};$$

and these substituted in (1), and the result cleared of radicals, give

$$(x^2+y^2)^2(a^2y^2+b^2x^2) = c^4x^2y^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the equation of the locus. In polar coordinates it becomes

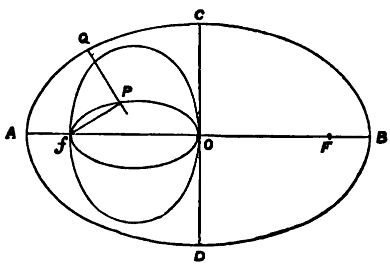
$$(a^2\sin^2\theta + b^2\cos^2\theta)\,r^2 = c^4\sin^2\theta\cos^2\theta.$$

The curve consists of 4 loops, one in each quadrant of the ellipse, and the whole area is

$$2\int_0^{\frac{1}{2}\pi} r^2 d\theta = 2c^4 \int_0^{\infty} \frac{z^2dz}{(b^2+a^2z^2)(1+z^2)^2},\ \text{where } z = \tan\theta,$$

$$= \tfrac{1}{2}\pi(a-b)^2.$$

2. Take the focus f as origin, and denote the point Q by $x'y'$; then the equations of the normal QP and perpendicular fP are



$$y-y' = \frac{a^2y'}{b^2(x'-c)}(x-x')\ \ldots (4),$$

and $$y = -\frac{b^2(x'-c)}{a^2y'}x \ldots (5).$$

From the last of these and $\frac{(x'-c)^2}{a^2} + \frac{y'^2}{b^2} = 1$, we get the values of x', y', and thence by (4) the equation of the locus of P is

$$(a^2y^2+b^2x^2)(y^2+x^2-cx)^2 = c^4x^2y^2,$$

which in polar coordinates becomes

$$(r-c\cos\theta)^2(a^2\sin^2\theta+b^2\cos^2\theta) = c^4\sin^2\theta\cos^2\theta,$$

or $$r = c\cos\theta \pm \frac{c^2\sin\theta\cos\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}}.$$

This curve consists of two loops, one within the other, as in the diagram, the inner being described while Q moves over the semi-periphery DAC, and the outer when Q moves over CBD.

Doubling for the whole loop, we have

$$\int_0^{\frac{1}{2}\pi} r^2 d\theta = \tfrac{1}{2}a(a-b)\,\pi \mp bc \pm a^2\sin^{-1}\left(\frac{c}{a}\right),$$

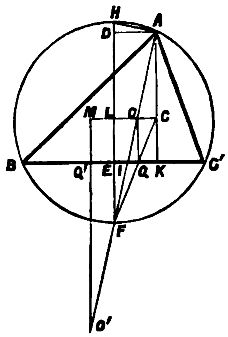
the upper sign applying to the outer loop, the lower to the inner. The whole area passed over by the radius vector fP is

$$\pi a(a-b) = \text{area of both loops.}$$

4655. (By T. T. Wilkinson.)—Let ABC be a triangle inscribed in a circle; O, O′ the centres of the inscribed and escribed circles touching the

base BC in Q, Q′; AK and AD perpendiculars upon BC, and a diameter HF bisecting BC in E; prove that HD . OQ = EQ . QK, and HD . O′Q′ =EQ . Q′K.

Solution by E. Rutter; H. Murphy; R. Tucker; *and others.*



Let AF meet EK in I, then it is readily seen that AI is cut harmonically in O, O′; and OO′ is bisected in F; therefore AO . OF = AF . OI, whence

$$EQ . QK = EK . IQ \text{ .... } (1).$$

Now, from similar triangles ADH, OIQ, O′IQ′, we have

$$HD . OQ = AD . IQ = EK . IQ = EQ . QK,$$

and $$\frac{O'Q'}{OQ} = \frac{IO'}{IO} = \frac{AO'}{AO} = \frac{Q'K}{QK};$$

therefore $HD . O'Q' = EQ . Q'K$.

4613. (By N'Imports.)—Required rational values of x that will make each of the expressions $400x^2+10x$, $400x^2+13x$, $400x^2+17x$ a rational square.

I. *Solution by* S. Tebay.

Consider the equation $a^2x^2+bx = z^2$, or $bx = z^2-a^2x^2$. Here we can take $z+ax = mb$, $z-ax = \dfrac{x}{m}$; and therefore $x = \dfrac{m^2b}{1+2ma}$. Let $m = 1$, then $x = \dfrac{b}{1+2a}$, and three simple values are $\dfrac{10}{41}$, $\dfrac{13}{41}$, $\dfrac{17}{41}$.

II. *Solution by* S. Bills.

Dividing each of the given expressions by 400, and then putting $\dfrac{10}{400}=a$, $\dfrac{13}{400} = b$, $\dfrac{17}{400} = c$, we shall have to find

$$x^2+ax = \square \text{ ... } (1),\quad x^2+bx = \square \text{ ... } (2),\quad x^2+cx = \square \text{ ... } (3).$$

Put $x^2+ax = (x-r)^2 = x^2-2rx+r^2$; from this we find $x = \dfrac{r^2}{2r+a}$. Substituting this in (2) and (3), we get

$$r^2+b(2r+a) = \square,\quad r^2+c(2r+a) = \square \text{ .... } (4, 5).$$

Assume $r^2+b(2r+a) = (r-s)^2 = r^2-2rs+s^2$; we find from this $r = \dfrac{s^2-ab}{2(s+b)}$. Substituting this value of r in (5), and reducing, we shall have to find $s^4+4cs^3+(4ac+4bc-2ab)s^2+4abcs+a^2b^2 = \square$ .... (6).

Assume (6) $= (s^2-2cs-ab)^2 = s^4-4cs^3+(4c^2-2ab)s^2+4abcs+a^2b^2$; we got from this $s = \frac{1}{2}(c-a-b)$. Restoring the values of a, b, c, we find $s = -\frac{3}{400}$; and thence, $r = -\frac{121}{8000}$ and $x = -\frac{14641}{336000}$. This value of x is *negative*; a *positive* value of x may be obtained as follows:—

Substituting for a, b, c in (6), the equation becomes

$$s^4+\frac{68}{400}s^3+\frac{1304}{400^2}s^2+\frac{8840}{400^3}s+\frac{16900}{400^4} = \square \text{ .... (7).}$$

In (7) put $s = t-\frac{3}{400}$; we then obtain

$$t^4+\frac{56}{400}t^3+\frac{822}{400^2}t^2+\frac{2744}{400^3}t+\frac{361}{400^3} = \square \text{ .... (8).}$$

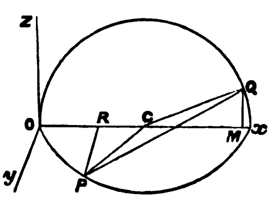
$$\text{Put (8)} = \left(t^2+\frac{28}{400}t+\frac{19}{400^2}\right)^2 = t^4+\frac{56}{400}t^3+\frac{822}{400^2}t^2+\frac{1064}{400^3}t+\frac{361}{400^4};$$

from this we find $t = \frac{21}{380}$, and $s = t-\frac{3}{400} = \frac{363}{7600}$; from this we shall obtain a *positive*, though rather large, value for x.

For the sake of symmetry, I have not reduced some of the fractions to their lowest terms.

4563. (By J. Freeman.)—Two rigid and equal semi-circular arcs of matter with uniform section and density are hinged together at both extremities. The matter attracts according to the law of gravitation. If equal and opposite forces applied along the line joining the middle points of the semicircles keep them apart with their planes at right angles, prove that the magnitude of each force will be $4m^2 \log(1+\sqrt{2})$, where m is the mass of unit length of arc.

Solution by Rev. J. R. Wilson.



Let P, Q be two elemental arcs, one on either semicircle. Let OCx be the common diameter, and draw Oy, Oz perpendicular to Ox, one in each of the two planes. Let OC$=c$, OCP$=\theta$, QC$x=\phi$. Draw PR, QM perpendicular to Ox. Then the attraction of P on Q is

$$\frac{mc\delta\theta \,.\, mc\delta\phi}{PQ^2}.$$

The resolved part of this attraction parallel to Oy is

$$\frac{m^2c^2\delta\theta\,\delta\phi}{PQ^2}\cdot\frac{PR}{PQ},$$

and the moment of this about Ox is $\frac{m^2c^2\delta\theta\,\delta\phi}{PQ^3}$. PR . QM.

Now $\qquad PR = c\sin\theta, \quad QM = c\sin\phi,$

and $PQ^2 = PR^2 + RM^2 + QM^2 = c^2 \sin^2\theta + (c\cos\theta + c\cos\phi)^2 + c^2\sin^2\phi$
$= 2c^2(1+\cos\theta\cos\phi)$.

Therefore the sum of the moments of all such resolved attractions taken throughout both semicircles is

$$\frac{m^2c}{2\sqrt{2}}\int_0^\pi\int_0^\pi \frac{\sin\theta\sin\phi}{(1+\cos\theta\cos\phi)^{\frac{3}{2}}}\,d\theta\,d\phi$$

$$= \frac{m^2c}{2\sqrt{2}}\int_0^\pi \frac{2\sin\theta}{\cos\theta}\left\{\frac{1}{(1-\cos\theta)^{\frac{1}{2}}} - \frac{1}{(1+\cos\theta)^{\frac{1}{2}}}\right\}d\theta$$

$$= \frac{m^2c}{\sqrt{2}}\int_0^\pi \frac{(1+\cos\theta)^{\frac{1}{2}}}{\cos\theta}\,d\theta - \frac{m^2c}{\sqrt{2}}\int_0^\pi \frac{(1-\cos\theta)^{\frac{1}{2}}}{\cos\theta}\,d\theta$$

$$= -\frac{m^2c}{\sqrt{2}}\log\frac{(1-\cos\theta)^{\frac{1}{2}}-1}{(1-\cos\theta)^{\frac{1}{2}}+1} + \frac{mc^2}{\sqrt{2}}\log\frac{(1+\cos\theta)^{\frac{1}{2}}-1}{(1+\cos\theta)^{\frac{1}{2}}+1},$$

between the limits 0 and π. It will be found that this reduces to

$$2m^2c\sqrt{2}\log(1+\sqrt{2}).$$

Hence, if P be the required force,

$$P.c\sin\tfrac{1}{2}\pi = 2m^2c\sqrt{2}\log(1+\sqrt{2});$$

therefore $$P = 4m^2\log(1+\sqrt{2}).$$

4629. (By R. F. Scott.)—Any point on the parabola $y^2 = 4ax$ being given by $y = \frac{2a}{m}n = \frac{a}{m^2}$; show that (1) the equation to the circle with respect to which the triangle formed by the tangents at $m_1m_2m_3$ is self-conjugate, is

$$x^2+y^2+2ax - \frac{2ay}{m_1m_2m_3}\left\{1+m_2m_3+m_3m_1+m_1m_2\right\}$$
$$+\frac{a^2}{m_1^2m_2^2m_3^2}\left\{m_1m_2+m_2m_3+m_3m_1+m_1m_2m_3(m_1+m_2+m_3) + m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2\right\} = 0,$$

and (2) the radical axis of the four circles corresponding to the triangles formed from the four points m_1, m_2, m_3, m_4 is

$$2y = a\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_4}\right).$$

Solution by the Proposer.

The equation of the tangent at the point m is $y = mx + \frac{a}{m}$.

The equation in trilinear coordinates of the circle with respect to which a triangle ABC is self-conjugate is $\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C = 0$; where $\alpha=0$, $\beta=0$, $\gamma=0$ are the equations of the sides.

Let the tangents at the points m_1, m_2, m_3 form the triangle.

We find $\sin 2A = \dfrac{2(m_2-m_3)(1+m_2m_3)}{(1+m_2^2)(1+m_3^2)}$; thus the equation to the circle becomes $\dfrac{\left(y-m_1x-\dfrac{a}{m_1}\right)^2}{1+m_1^2}\,\dfrac{(m_2-m_3)(1+m_2m_3)}{(1+m_2^2)(1+m_3^2)}+\ldots=0,$

or $(x^2+y^2)\left\{m_2m_3(m_2-m_3)+\ldots\right\}-2ay\left\{\dfrac{(m_2-m_3)(1+m_2m_3)}{m_1}+\ldots\right\}$

$$+2ax\left\{m_2m_3(m_2-m_3)+\ldots\right\}+a^2\left\{\frac{(m_2-m_3)(1+m_2m_3)}{m_1^2}+\ldots\right\}=0$$

the missing terms being given by symmetry.

Let $D=\begin{vmatrix}1 & 1 & 1\\ m_1 & m_2 & m_3\\ m_1^2 & m_2^2 & m_3^2\end{vmatrix}$; then the coefficient of x^2+y^2 is $-D$,

of $\dfrac{-2ay}{m_1m_2m_3}$ is $\Sigma m_2m_3(m_2-m_3)+\Sigma m_2^2m_3^2(m_2-m_3)$

$$=-D\left\{1+m_1m_2+m_2m_3+m_3m_1\right\},$$

of $\dfrac{a^2}{m_1^2m_2^2m_3^2}$ is $\Sigma m_2^2m_3^2(m_2-m_3)+\Sigma m_2^3m_3^3(m_2-m_3)$

$$=-D\left\{m_1m_2+m_2m_3+m_3m_1+m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2+m_1m_2m_3(m_1+m_2+m_3)\right\},$$

and the equation of the circle is that given in the question.

The equation of the radical axes of the circles corresponding to $m_1m_2m_3$ and $m_2m_3m_1$ is

$$2ay\left\{\frac{1}{m_2m_3m_4}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_4}-\frac{1}{m_1m_2m_3}-\frac{1}{m_1}-\frac{1}{m_2}-\frac{1}{m_3}\right\}$$

$$+a^2\left\{\frac{1}{m_1m_2m_3}\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}\right)+\frac{1}{m_1^2}+\frac{1}{m_2^2}+\frac{1}{m_3^2}+\frac{m_1+m_2+m_3}{m_1m_2m_3}\right.$$

$$\left.-\frac{1}{m_2m_3m_4}\left(\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_4}\right)-\frac{1}{m_2^2}-\frac{1}{m_3^2}-\frac{1}{m_4^2}+\frac{m_2+m_3+m_4}{m_2m_3m}\right\},$$

which divides by $a\left(\dfrac{1}{m_2m_3}+1\right)\left(\dfrac{1}{m_4}-\dfrac{1}{m_1}\right)$, and the equation is sa given above.

4654. (By S. A. Renshaw.)—If R, Q, F, F′ denote respectively a right angle, the angle contained by a pair of tangents to a conic, and the angles subtended at the foci of the conic; prove that

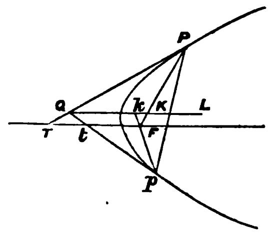
in the parabola $F=2Q$ .... (1),

in the ellipse $F+F'=2(2R-Q)=2$ (supplement of Q) .... (2),

in the hyperbola $F+F'=2Q$ .... (3).

Solution by the PROPOSER; R. TUCKER; J. F. WILSON; *and others.*

1. Draw OKL parallel to the axis, and meeting FP in K, and let PQ meet the axis T. Then, since FP = FT,



therefore $\quad KP = KQ.$

Hence the exterior angle $PKL = 2PQK.$

Similarly, if pF be produced to meet QL in k, $\quad Lkp = 2Ftp,$

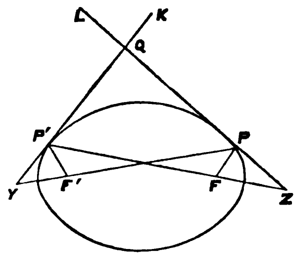
and therefore

$$PKL + pkL = 2PQK + 2pQK;$$

that is,

$$PFp = 2PQp, \quad \text{or} \quad F = 2Q.$$

2. Produce P′Q, PQ to meet PF′, P′F in Y and Z, and also beyond Q.



Then

$$KQP = QYP + QPY$$
$$= QYP + FPZ;$$

similarly,

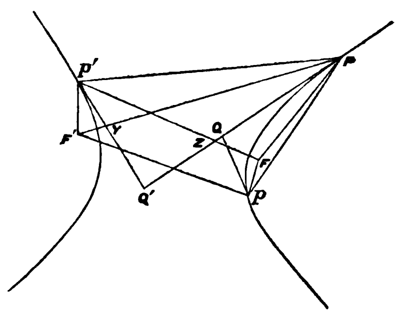
$$KQP = QZP' + FP'Y;$$

therefore

$$2KQP = P'YF' + F'P'Y + PZF + FPZ$$
$$= PF'P' + PFP',$$

or $\quad F + F' = 2(2R - Q).$

3. First, when the tangents touch the same branch of the hyperbola. Calling the angles contained by the focal distances of P and p, P and p respectively,



evidently, $\quad F = Q + \frac{1}{2}P + \frac{1}{2}p,$

and $\quad F' = Q - \frac{1}{2}P - \frac{1}{2}p,$

therefore $\quad F + F' = 2Q.$

Next, when the tangents at P and p' touch opposite branches of the hyperbolas. Then, putting Y and Z for the angles $p'ZQ'$ and PYQ' respectively,

$$Q' = 2R - Z - \frac{p'}{2}, \quad \text{and} \quad Q' = 2R - Y - \frac{p}{2};$$

therefore $\quad 2Q' = \left\{2R - Y - \frac{P'}{2}\right\} + \left\{2R - Z - \frac{P}{2}\right\} = F' + F.$

4330. (By J. J. WALKER.)—Through the angles A, B, C of a spherical triangle arcs of great circles are drawn which co-intersect in the

point O, and meet the opposite sides in the points D, E, F respectively: prove that $\frac{\sin AO}{\sin OD} = \frac{\sin AE}{\sin CE} \cos CD + \frac{\sin AF}{\sin BF} \cos BD$ .... (1);

$$\frac{\sin OD}{\sin AD} \cos AO + \frac{\sin OE}{\sin BE} \cos BO + \frac{\sin OF}{\sin CF} \cos CO = 1 \;\ldots\; (2).$$

Solution by the PROPOSER.

1. From the sides of the triangle ACD, cut by the transversal BOE,

$$\frac{\sin AO}{\sin OD} = \frac{\sin AE}{\sin EC} \cdot \frac{\sin CB}{\sin BD} = \frac{\sin AE}{\sin EC} \cos CD + \frac{\sin AE}{\sin EC} \cdot \frac{\sin CD}{\sin DB} \cos BD$$

$$= \frac{\sin AE}{\sin EC} \cos CD + \frac{\sin AF}{\sin FB} \cos BD.$$

2. Applying the relation just proved to the triangle COD, the sides of which are divided in D, E, F by the three concurrent arcs DA, CA, BA,

we have $\frac{\sin OE}{\sin BE} \cos BD + \frac{\sin OF}{\sin CF} \cos CD = \frac{\sin OA}{\sin AD}$,

therefore $\frac{\sin OE}{\sin BE} \cos BD \cos OD + \frac{\sin OF}{\sin CF} \cos CD \cos OD$

$$= \frac{\sin OA \cos OD}{\sin AD} = 1 - \frac{\sin OD}{\sin AD} \cos AO.$$

Now $\cos BD \,.\, \cos OD = \cos BO - \sin BD \sin OD \cos ODB$

and $\cos CD \,.\, \cos OD = \cos CO + \sin CD \sin OD \cos ODB$.

Substituting these values, and observing that

$$\frac{\sin OE \,.\, \sin BD}{\sin BE} = \frac{\sin OF \,.\, \sin CD}{\sin CF},$$

there results $\frac{\sin OD}{\sin AD} \cos AO + \frac{\sin OE}{\sin BE} \cos BO + \frac{\sin OF}{\sin CF} \cos CO = 1.$

4700. (By the EDITOR.)—Show that (1) the average area of all the circles that can be drawn touching a given circle, and having their centres equably distributed over its area, is one-sixth of the area of the circle; and (2) if a polygon circumscribes a circle, the average area of all the circles that lie wholly within the polygon, touch its perimeter, and have their centres equably distributed over its area, is also one-sixth of the area of the circle.

I. *Solution by* E. B. ELLIOTT.

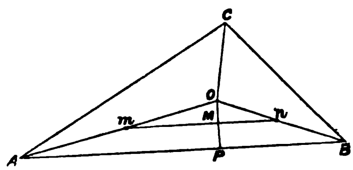
Let a be the radius of the given circle; then the ratio of the number of centres of the touching circles that lie in the ring between the central

distances r and $r+dr$ to the whole number is $\frac{2\pi r dr}{\pi a^2} = \frac{2rdr}{a^2}$. The area of each of these circles is $\pi(a-r)^2$. Thus the mean value of the area is

$$\int_0^a \pi(a-r)^2 \frac{2rdr}{a^2} = \frac{2\pi}{a^2}\int_0^a (a-r)^2 rdr = \frac{2\pi}{a^2}\int_0^a (a-r)r^2 dr = \tfrac{1}{6}\pi a^2.$$

II. *Solution by* C. B. S. Cavallin.

We begin to solve the question in the case of a triangle ABC. From the centre O of the inscribed circle draw OA, OB, OC. It is obvious that the perpendicular on AB from any point in the area OAB is less than the perpendiculars from the same point on the sides AC and BC;



and also that the perpendicular on AB from any point in the area AOBC is greater than the perpendiculars from the same point on AC and BC. Thus every circle contemplated in the question which touches AB has its centre in the area AOB.

Draw the line mn parallel to AB, terminating in OA and OB, and draw OP perpendicular to AB and cutting mn in M. Put OM $= x$, OP (the radius of the inscribed circle) $= r$, and a, b, c for the sides of the triangle.

The area of a circle having its centre on mn, and touching AB, is thus $\pi(r-x)^2$. But as the circles are to have their centres equably distributed over the area ABC, these centres must also be equably distributed along mn. Thus the sum of the areas of the circles having their centres on mn is

$$mn \,.\, \pi(r-x)^2 = \frac{c\pi}{r}\, x(r-x)^2.$$

Consequently the sum of the areas of the circles having their centres in OAB is

$$\frac{c\pi}{r}\int_0^r x(r-x)^2 dx = \frac{c\pi}{r}\int_0^r x^2(r-x)\,dx = \tfrac{1}{12}\pi r^3 c.$$

We conclude therefore that the sum of the areas of the circles having their centres in ABC is $\frac{1}{12}\pi r^3(a+b+c)$; and the number of the circles being $\frac{1}{2}r(a+b+c)$, we find that the sought average area is $\frac{1}{6}\pi r^2$.

By similar reasoning with regard to a polygon circumscribing a circle we get the same result. Thus assuming a regular polygon circumscribing a circle, and increasing the number of the sides indefinitely, we see that the same average area is obtained for a circle.

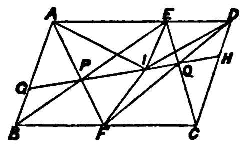
If, instead of supposing that the centres are equably distributed over the *area*, we suppose that their distribution is equable along the *perimeter* of a polygon similar to the given one, and touching a circle concentric with the circle inscribed in the given polygon, and that the density of this distribution is proportional to the positive nth power of the radius of the concentric circle, we easily obtain, in this case, for the average area

$$\frac{2\Gamma(n+3)}{\Gamma(n+5)}\,\pi r^2.$$

By putting in the last formula $n=0$, we get, as we ought, $\frac{1}{6}\pi r^2$.

2018. (By W. Hopps.)—E, F, are any two points in the opposite sides AD, BC of a parallelogram, and P, Q the intersections of the lines AF, BE, and DF, CE; prove that the line passing through P, Q bisects the parallelogram.

Solution by the Proposer.



Let G, H be the points at which the line PQ meets AB, CD; draw EI parallel to CD, and join AI, DI, FI. Then, by similar triangles, we have

$$PG : PI = PB : PE = PF : PA;$$

therefore $\triangle$APG = FPI (Euc vi. 15); therefore $\triangle$AGI = AFI; and similarly $\triangle$DHI = DFI. Hence, fig. AGHD (= $\triangle$AID + AGI + DHI = AID + AFI + DFI) = $\triangle$AFD = $\frac{1}{2}$ parallelogram ABCD; that is, GH *bisects* the parallelogram.

Note.—As every line bisecting a parallelogram passes through the intersection of the diagonals, it is obvious that this Question is a particular case of Prop. 139, bk. vii., of Pappus's *Mathematical Collections*. But, unlike that proposition, it admits of a geometrical solution independent of the doctrine of *Harmonicals*, as appears from what precedes.

3614. (By Dr. Hart.)—Find general rational expressions for a, b, x, that will make $4x^4 + 4xa^3 + 4bx + ab = 0$.

Solution by the Proposer.

This equation may be put under the form of $(2x^2 + ax)^2 - a^2x^2 + 4bx + ab = 0$; whence, by transposing, $(2x^2 + ax)^2 = a^2x^2 - 4bx - ab$ .... (1), where $a^2x^2 - 4bx - ab$ must be a square. Put it $= (ax - b)^2$; then, reducing, we find $x = \dfrac{a+b}{2a-4}$; substituting in (1) and extracting square root, we have

$$\frac{2(a+b)^2}{(2a-4)^2} + \frac{a(a+b)}{2a-4} = \frac{a^2 - ab + 4b}{2a-4};$$

whence, by transposing, we have $\dfrac{2(a+b)^2}{(2a-4)^2} = -b$;

clearing fractions, &c., $a^2 + 2ab + b^2 = -2a^2b + 8ab - 8b$; transposing, &c., we have $(2b+1)a^2 - 6ab = -b^2 - 8b$; dividing by $2b+1$, we have

$$a^2 - \frac{6b}{2b+1} \cdot a = -\frac{b^2 + 8b}{2b+1},$$

therefore $\quad a = \dfrac{3b \pm (b+2)\sqrt{-2b}}{2b+1}$, and $\quad x = \dfrac{2b \pm \sqrt{-2b}}{2(-1) \pm \sqrt{-2b}}$.

Now, to get rid of the imaginary quantity $\sqrt{-2b}$, put $b=-\frac{1}{2}m^2$,

then we have $a=\dfrac{-3m^2\pm(m^2+4m)}{2(-m^2+1)}$, and $x=\dfrac{-m^2\pm m}{2(-1\pm m)}$,

where m may be any number.

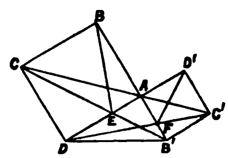
If $m=1$, then, using the positive sign, $b=-\frac{1}{2}$, $a=\frac{0}{0}$, $x=\frac{0}{0}$, and the terms of the given equation are $\frac{0}{0}+\frac{0}{0}-\frac{0}{0}-\frac{0}{0}$, whose sum is $=0$. It is a singular fact, that, if the negative sign be used, $b=-\frac{1}{2}$, $a=-\frac{6}{0}=-\infty$, $x=\frac{1}{2}$; and the terms are $\frac{1}{4}-\infty-1+\infty$, whose sum is $-\frac{3}{4}$, but which ought to be $=0$.

If $m=2$, then, using the positive sign, $b=-2$, $a=2$, $x=-1$, and the terms are $4-8+8-4$, whose sum is $=0$. Using the negative sign, $b=-2$, $a=2$, $x=1$, and the terms are $4+8-8-4$, whose sum is $=0$.

If $m=3$, then, using the positive sign, $b=-\frac{9}{2}$, $a=\frac{21}{8}$, $x=-\frac{3}{2}$, and the terms are $\frac{81}{4}-\frac{567}{16}+27-\frac{189}{16}=0$. Using the negative sign, $b=-\frac{9}{2}$, $a=\frac{3}{4}$, $x=\frac{3}{2}$, and the terms are $\frac{81}{4}+\frac{81}{8}-27-\frac{27}{8}=0$, &c. &c.

4447. (By N'Importe.)—ABCD, AB′C′D′ are two squares, BAB′, DAD′ being straight lines, B′C meets AD in E and C′D meets AB′ in F: prove that AE and AF are equal to each other.

Solution by H. Murphy, A. P. Shepherd, *and others.*



Draw EB, FD′, CAC′, then plainly we have

$\triangle DD'F = DAC' = DAB'$,

and in like manner

$\triangle BEB' = DAB'$; therefore $\triangle DD'F = BEB'$,

and their bases DD′ and BB′ are equal, hence the altitudes FA and EA are equal. Otherwise: AE : AB′ = BC : BB′ = DA : DD′ = AF : C′D′; but AB′ = C′D′, therefore AE = AF.

Cor. 1.—These lines are equal to the side of a square inscribed in the triangle DAB′ on the right angle A; for $(b+p)s = 2\triangle DAB' = DD'.FA$; therefore $s = FA$, where s = side of square, $p = AD$, and $b = AB'$.

Cor. 2.—It may be also shewn that if C′D′ and CB were produced and their intersection joined to A, this line would be perpendicular to DB′.

4229. (By N'Importe.)—Find rational plane triangles, whose sides are integers, such that two of the sides shall differ by unity only; and show that 17, 25, 26 are sides of *one* such triangle.

Solution by HENRY STANLEY MONCK.

The following Solution is only accommodated to the particular case given in the Question, but it enables me to prove some interesting properties of the series for Pythagorean triangles.

Taking the triangle 26, 25, 17, the ordinary expression for the area $\sqrt{s(s-a)(s-b)(s-c)}$ becomes $\sqrt{34 \,.\, 8 \,.\, 9 \,.\, 17}$. Comparing this with the series

$$\begin{array}{cccccc} 3 & 8 & 20 & 49 & 119 & 288 \\ 5 & 12 & 29 & 70 & 169 & 408 \\ 4 & 9 & 21 & 50 & 120 & 289, \end{array}$$

I find $s-a$ and $s-b$ are respectively equal to the upper and lower figures of the second column and $s-c$ equal to their sum. Now I say that similar triangles can be formed from the fourth, sixth, and every other even column, by putting $s-a =$ the top figure, $s-b =$ the lower figure, and $s-c =$ their sum. The sides of the first triangle thus found are 148, 149, 99, of the second 866, 867, 577, &c.

The peculiarity of the figures 34, 8, 9, 17 is that the third is a perfect square, the second half a perfect square (or twice a perfect square, since four times a square is always a square), and the first twice the fourth. Hence the product makes a perfect square. Now I shall show that, if in any of the above even columns, a, b, c are of the form $2a^2b\gamma^2$, the next column but one will also have its upper and lower figures of the form $2\square$ and $\square$. Their position will indeed be inverted, but that is immaterial as regards the continuation of the process.

Let us then take the columns,

$$\begin{array}{lll} 2a^2 & 2a^2+b & 4a^2+2b+\gamma^2 \\ b & 2a^2+b+\gamma^2 & 4a^2+3b+2\gamma^2 \\ \gamma^2 & b+\gamma^2 & 2a^2+2b+2\gamma^2. \end{array}$$

Here, since the intermediate column represents a Pythagorean triangle, we have

$$(2a^2+b)^2+(b+\gamma^2)^2 = (2a^2+b+\gamma^2)^2 \;\; 4a^4+4a^2b+2b^2+2b\gamma^2+\gamma^4$$
$$= 4a^4+4a^2b+b^2+4a^2\gamma^2+2b\gamma^2+\gamma^4.$$

Whence at once $b^2 = 4a^2\gamma^2$ or $b = 2a\gamma$. Putting $2a\gamma$ for b in the third column, we find the top figure is $(2a+\gamma)^2$, and the bottom figure is $2(a+\gamma)^2$, which is the same form as before; and as the difference of the top and bottom figures is constant, we have all the requisites for forming a triangle of the kind required in the Question. We have only to put

$$s-a' = (2\alpha+\gamma)^2 \; s-b = 2(\alpha+\gamma)^2,$$

and $s-c' = (2\alpha+\gamma)^2+2(\alpha+\gamma)^2$, and determine a', b', c' accordingly. Then

$$s(s-a')(s-b')(s-c') = (2\alpha+\gamma^2) \,.\, 4(\alpha+\gamma)^2 \,.\, \left\{2(\alpha+\gamma)^2+(2\alpha+\gamma)\right\}^2,$$

which gives a rational area.

This result affords a simpler mode of formation for the successive triangles. Since in the triangle 26, 25, 17, $s-a = 2 \,.\, 2^2$, and $s-b = 3^2$, we may form our series thus,

$$\begin{array}{ccccc} 2 & 5 & 12 & 29 & 70 \\ 3 & 7 & 17 & 41 & 99, \text{ \&c.} \end{array}$$

making in each case the top figure equal to $\alpha+\gamma$ and the bottom figure $2\alpha+\gamma$, where α is the top figure and γ the bottom figure of the preceding column. Then, by putting

$$s-a = 2\alpha_n^2, \quad s-b = \gamma_n^2, \quad \text{and} \quad s-c = 2\alpha_n^2+\gamma_n^2,$$

we obtain a triangle in continuation of the series.

I now venture to generalise the result arrived at above with regard to the series. It is already evident that every even column in that series is of the form $\begin{matrix} 2\alpha^2 \\ 2\alpha\beta \\ \beta^2 \end{matrix}$ or $\begin{matrix} \beta^2 \\ 2\alpha\beta, \\ \alpha^2 \end{matrix}$ and the odd columns which give Pythagorean triangles are of the form

$$\begin{matrix} 2\alpha^2 + 2\alpha\beta \\ 2\alpha^2 + 2\alpha\beta + \beta^2 \\ 2\alpha\beta + \beta^2. \end{matrix}$$

The reader will easily verify for himself that, if α and β be any positive integers whatever, the square of the middle figure here is equal to the sums of the squares of the upper and lower ones. I wish, however, to show that if we introduce negative signs into the series, these results are still unaffected.

I have shown that from the column $\begin{matrix} 3 \\ 5 \\ 4 \end{matrix}$ three really distinct second columns can be obtained, viz., by taking all three positive, taking 3 negative, and taking 4 negative. They are

8	which answers	8	and	2
12	the triangle	4		6
9	in the Question	1		9.

As the first of these has for the top figure $2 \cdot 2^2$, and the bottom figure 3^2, so the second has for its top figure $2 \cdot 2^2$ and the bottom figure 2^2, and the third has for a top figure $2 \cdot 1^2$, and a bottom figure 3^2. They are thus all similar in form. I now proceed to show that the form cannot be altered by introducing a negative sign (in the top or bottom figure of an odd or Pythagorean column) at any subsequent stage. Take the column

$$\begin{matrix} 2\alpha^2 & 2\alpha^2 + 2\alpha\beta & \beta^2 \\ 2\alpha\beta & 2\alpha^2 + 2\alpha\beta + \beta^2 & 2\alpha\beta + 2\beta^2 \\ \beta^2 & 2\alpha\beta + \beta^2 & 2\alpha^2 + 4\alpha\beta + 2\beta^2, \end{matrix}$$

and let the third column be formed by taking the top figure of the second column negatively as above, and we have plainly the same form as before, viz., $\begin{matrix} \beta^2 \\ 2\beta(\alpha+\beta) \\ 2(\alpha+\beta)^2 \end{matrix}$; similarly, if we had taken $2\alpha\beta + \beta^2$ of the second column negatively, our third column would be

$$\begin{matrix} 4\alpha^2 + 4\alpha\beta + \beta^2 \\ 4\alpha^2 + 4\alpha\beta \\ 2\alpha^2 \end{matrix} = \begin{matrix} (2\alpha+\beta)^2 \\ (2\alpha+\beta) \\ 2\alpha^2 \end{matrix}$$

which is still of the same form. Hence I conclude

1. That however we operate on the elementary triangle 3, 5, 4 by the rule in Question 4102, whether introducing negative signs or not, the even columns are always of the form $\begin{matrix} \alpha^2 \\ 2\alpha\beta \\ 2\beta^2 \end{matrix}$ or $\begin{matrix} 2\alpha^2 \\ 2\alpha\beta. \\ \beta^2 \end{matrix}$ And the odd columns are always of the form $\begin{matrix} 2\alpha(\alpha+\beta) \\ 2\alpha^2 + 2\alpha\beta + \beta^2 \\ (2\alpha+\beta)\beta \end{matrix}$ or $\begin{matrix} \alpha(\alpha+2\beta) \\ 2\alpha^2 + 2\alpha\beta + \beta^2 \\ 2\beta(\alpha+\beta) \end{matrix}$

2. That, α and β being any two whole numbers, $(2\alpha^2 + 2\alpha\beta)$, $(2\alpha^2 + 2\alpha\beta + \beta^2)$, and $(2\alpha\beta + \beta^2)$ form the three sides of a Pythagorean triangle.

3. Since all Pythagorean triangles can be formed by applying the

process (with negative signs when necessary) to the elementary triangle 3, 5, 4, we can obtain by this process all possible values $(2\alpha^2+2\alpha\beta)$, $(2\alpha^2+2\alpha\beta+\beta^2)$, and $(2\alpha\beta+\beta^2)$, in which α and β are whole numbers.

4. All possible Pythagorean triangles can be found by giving all possible integral values to α and β in the expressions $(2\alpha^2+2\alpha\beta)$, $(2\alpha^2+2\alpha\beta+\beta^2)$, and $(2\alpha\beta+\beta^2)$.

These results are only subject to qualification in respect of what I may call multiple triangles, such as 6, 10, 8; 24, 26, 10; 9, 15, 12.

[See *Reprint*, Vol. XX., pp. 20, 54, 75, 76, 87, 97, 99, 100.]

3156. (By A. B. Evans.)—Find a positive integral value of x that will satisfy the condition $94x^2+57x+34=\square$.

I. *Solution by* Artemas Martin.

From the given expression subtract $(3x-4)^2$, and there remains

$$85x^2+81x+18, = (5x+3)(17x+6);$$

therefore $\quad 94x^2+57x+34 = (3x-4)^2+(5x+3)(17x+6) = \square,$

which put $\qquad = \left\{(3x-4)-\frac{m}{n}(5x+3)\right\}^2;$

then, reducing, we find $\quad x = \dfrac{3m^2+8mn-6n^2}{17n^2+6mn-5m^2}.$

In order to have x integral, put

$$17n^2+6mn-5m^2=1, \text{ hence } 5m = 3n \pm (94n^2-5)^{\frac{1}{2}},$$

therefore $94n^2-5$ must be a square, which is the case when $n=3$; therefore $m=-4$, and $x=-102$.

To find a *positive* value of x, put $x=y-102$, then by substitution the original expression becomes

$$94y^2-19119y+972196=\square.$$

Subtract $(2y-986)^2$, and the remainder is

$$90y^2-15175y = 5y(18y-3035),$$

and $\quad 94y^2-19119y^2+972196 = (2y-986)^2+5y(18y-3035) = \square.$

Assume the right-hand member

$$= \left\{(2y-986)-\frac{p}{q}(18y-3035)\right\}^2.$$

Involving and reducing, $\quad y = \dfrac{1972pq-3035p^2}{5q^2+4pq-18p^2}.$

Now put $\quad 5q^2+4pq-18p^2=-1$, then $y = 3035p^2-1972pq$, and $q = \frac{1}{5}(-2p\pm\sqrt{94p^2-5}), = -7$, by taking $p=3$, and using the lower sign. Therefore $y = 68727$ and $x = 68625$.

II. *Solution by* Dr. HART.

The general expression $ax^2+bx+c=\square$, when $b^2-4ac=\square$, can be divided into two binomial factors, and thus can be solved. But if it cannot be so divided, subtract a binomial square from it; and if, in the remainder, the same relation between the quantities a, b, c exists, that remainder can be divided into two factors. Since, in $94x^2+57x+34$, the above relation does not exist, subtract $(3x-4)^2$, and the remainder is $85x^2+81x+18$. Let this be divided into the factors $5x+3$ and $17x+6$,

$$\text{then } 94x^2+57x+34=(3x-4)^2+(5x+3)(17x+6)=\square$$
$$=\left\{(3x-4)-\frac{m}{n}(5x+3)\right\}^2.$$

Reducing, we find $x=\dfrac{3m^2+8mn-6n^2}{17n^2+6mn-5m^2}$. In order to have x integral,

put $\qquad 17n^2+6mn-5m^2=1$, whence $m^2-\frac{6}{5}mn=\dfrac{17n^2-1}{5}$;

and $m=\frac{1}{5}(3n\pm\sqrt{94n^2-5})$, where $94n^2-5$ must be a $\square$, which is so when $n=3$, therefore $m=-4$, and $x=-102$. Here, to have a position value of x, put $x=y-102$; then by substitution

$$94x^2+57x+34=94y^2-19119y+992196=\square.$$

Put this $=\left(986-\dfrac{p}{q}y\right)^2$; then, reducing, we find $y=\dfrac{(1972p-19119q)q}{p^2-94q^2}$.

To find $p^2-94q^2=1$, develope $\sqrt{94}$ into a continued fraction, and the partial quotients will be 9; 1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1, 18, constituting one period, not reckoning 9; then form a series of converging fractions, and that fraction standing under 18, in an even place, is $\dfrac{2143295}{221064}$. Calling the numerator p and the denominator q, we shall have $y=12185931936$, whence $x=12185931834$, and innumerable other values of x may be found by means of the successive periods of quotients obtained by developing $\sqrt{94}$ into a continued fraction, or else by using the method laid down in my solution of Question 2981. [*Reprint*, Vol. XX., p. 64.]

4693. (By J. J. SYLVESTER.)—Show that any triangle may be made to move with its angles continually in contact with three circles, and the same with three straight lines, and the same (if isosceles) with two straight lines and a circle; but that no triangle can be made to move with its angles in contact with two circles and one straight line.

Solution by R. W. GENESE.

Prof. Sylvester has shown (*Transactions of London Mathematical Society*, No. 77, p. 35) that if a quadrilateral ABCD (in which AB = AD and CB = CD) is deformed, AB being fixed, then any point M rigidly connected

with BC will describe a pedal of a conic. Now, by properly choosing M, this pedal will reduce to a point circle and a real circle; thus the triangle MBC moves with its vertices on three circles. The pedal (a bicircular quartic) could never have a line factor; thus a triangle could not be found to move with its vertices on two circles and a straight line. Let a triangle MBC move with two of its vertices on fixed lines OB, OC. Since $\angle$ COB and BC are invariable, the circle circumscribing OBC has a constant radius. Therefore its centre P describes a circle centre O; thus an isosceles triangle can move with its vertices on two straight lines and a circle. Lastly, if angle CMB be equal to COB or its supplement, M will describe a straight line through O, for angle MOB = angle MCB = constant; therefore &c.

4694. (By Professor Cayley.)—Taking F, F′ a pair of reciprocal points in respect to a circle, centre O; then if F, F′ are centres of force, each force varying as (distance)$^{-n}$, prove that (1) the resultant force upon any point P on the circle is in the direction of a fixed point S on the axis OFF′; and if, moreover, the forces at the unit of distance are as $(\mathrm{OF})^{\frac{1}{2}(n-1)}$ to $(\mathrm{OF}')^{\frac{1}{2}(n-1)}$, then (2) the resultant force is proportional to

$$(\mathrm{SP})^{-\frac{1}{2}(n-1)} \,.\, (\mathrm{PV})^{-\frac{1}{2}(n+1)},$$

where PV is the chord through S.

Solution by R. W. Genese.

1. Resolving at right angles to PS, we get (if $\mathrm{FP} = r$, $\mathrm{F'P} = r'$)

$$\frac{\mu}{r^n} \sin \mathrm{FPS} = \frac{\mu'}{r'^n} \sin \mathrm{F'PS};$$

therefore
$$\frac{\mu}{r^n}\,\frac{2 \text{ triangle FPS}}{\mathrm{FP}\,.\,\mathrm{PS}} = \frac{\mu'}{r'^n}\,\frac{2 \text{ triangle F'PS}}{\mathrm{F'P}\,.\,\mathrm{P'S}};$$

therefore
$$\frac{\mu}{r^{n+1}}\,\mathrm{FS} = \frac{\mu'}{r'^{n+1}}\,\mathrm{F'S},$$

therefore
$$\frac{\mathrm{FS}}{\mathrm{F'S}} = \frac{r^{n+1}}{r'^{n+1}}\,\frac{\mu'}{\mu} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Now it is a property of the points F, F′ that $r : r' =$ constant; therefore S is a fixed point.

2. Again, let $\mathrm{OF} = a$, radius of circle $\mathrm{OA} = c$; then $\mathrm{OF}' = \dfrac{c^2}{a}$;

therefore
$$\frac{r}{r'} = \frac{\mathrm{FA}}{\mathrm{F'A}} = \frac{c-a}{a^{-1}c^2 - c} = \frac{a}{c}, \qquad \frac{\mathrm{OF}}{\mathrm{OF'}} = \frac{a}{a^{-1}c^2} = \frac{a^2}{c^2};$$

therefore
$$\frac{\mu}{\mu'} = \frac{a^{n-1}}{c^{n-1}},$$

therefore
$$\frac{\mathrm{FS}}{\mathrm{F'S}} = \frac{a^{n+1}}{c^{n+1}} \cdot \frac{c^{n-1}}{a^{n-1}} = \frac{a^2}{c^2}.$$

Now, resolving along PS, we have

$$\text{resultant force} = \frac{\mu}{r^n}\cos \mathrm{FPS} + \frac{\mu'}{r'^n}\cos \mathrm{F'PS}$$

$$\propto \frac{1}{r^n}\left(a^{n-1}\cos \mathrm{FPS} + \frac{c^{n-1}\,.\,a^n}{c^n}\cos \mathrm{F'PS}\right)$$

$$\propto \frac{1}{r^n}\left(c\cos \mathrm{FPS} + a\cos \mathrm{F'PS}\right).$$

But if $\mathrm{PS} = \rho$, $r\cos \mathrm{FPS} = \rho - \mathrm{FS}\cos \mathrm{PSF}$, $r'\cos \mathrm{SPF'} = \rho + \mathrm{F'S}\cos \mathrm{PSF'}$;

therefore $\quad c^2 r\cos \mathrm{FPS} + a^2 r'\cos \mathrm{SPF'} = \rho\,(c^2+a^2),$

$$cr\,(c\cos \mathrm{FPS} + a\cos \mathrm{F'PS}) = \rho\,(c^2+a^2).$$

Thus $\quad$ resultant force $\propto \dfrac{\rho}{r^{n+1}}$ .... (2).

Again $\quad r^2 = \rho^2 + \mathrm{FS}^2 - 2\rho\,.\,\mathrm{FS}\cos \mathrm{PSF},$

$$r'^2 = \rho^2 + \mathrm{F'S}^2 - 2\rho\,.\,\mathrm{F'S}\cos \mathrm{PSF'};$$

therefore $\quad c^2r^2 + a^2r'^2 = (c^2+a^2)\,\rho^2 + c^2\,.\,\mathrm{FS}^2 + a^2\,.\,\mathrm{F'S}^2.$

But $\quad \mathrm{FF'} = \dfrac{c^2}{a} - a = \dfrac{c^2-a^2}{a} = \dfrac{p}{a}$ say;

therefore $\quad \mathrm{FS'} = \dfrac{a^2}{c^2+a^2}\cdot\dfrac{p}{a},\quad \mathrm{F'S'} = \dfrac{c^2}{a^2+c^2}\cdot\dfrac{p}{a};$

therefore, since $c^2r^2 = a^2r'^2$,

$$2c^2r^2 = (c^2+a^2)\,\rho^2 + \left\{c^2\frac{a^4}{(c^2+a^2)^2} + a^2\frac{c^4}{(c^2+a^2)^2}\right\}\frac{p^2}{a^2}$$

$$= (c^2+a^2)\left\{\rho^2 + \frac{c^2p^2}{(c^2+a^2)^2}\right\}.$$

But $\quad \mathrm{OS} = a + \dfrac{pa}{c^2+a^2} = \dfrac{2ac^2}{c^2+a^2};$

therefore $\quad c^2 - \mathrm{OS}^2 = c^2 - \left\{\dfrac{2ac^2}{(c^2+a^2)}\right\}^2 = \dfrac{c^2p^2}{(c^2+a^2)^2} = \mathrm{SP}\,.\,\mathrm{SV};$

therefore $\quad r^2 \propto \rho^2 + \rho\,.\,\mathrm{SV} \propto \rho\,.\,\mathrm{PV}.$

Thus, from (2), we have

$$\text{resultant force} \propto \frac{\mathrm{SP}}{(\mathrm{SP}\,.\,\mathrm{PV})^{\frac{1}{2}(n+1)}} \propto (\mathrm{SP})^{-\frac{1}{2}(n-1)}\,.\,\mathrm{PV}^{-\frac{1}{2}(n+1)}.$$

4696. (By Prof. Clifford.)—Six circles pass through twelve points on a conic in the following order,

$(a)\ldots \mathrm{A_1A_2A_3A_4}$, $\quad(b)\ldots \mathrm{B_1B_2B_3B_4}$, $\quad(c)\ldots \mathrm{C_1C_2C_3C_4}$,

$(d)\ldots \mathrm{A_1A_2B_3C_4}$, $\quad(e)\ldots \mathrm{B_1B_2C_3A_4}$, $\quad(f)\ldots \mathrm{C_1C_2A_3B_4}$;

prove that two circles and another point may be taken arbitrarily, and that the circles *abc* meet the circles *def* in six new points which lie on the circumference of another circle.

Solution by F. D. Thomson.

For convenience let A_1, A_2, &c. denote the eccentric angles of the points A_1, A_2, &c.

Then, if the two circles (a) and (b) are given, we have

$$A_1 + A_2 + A_3 + A_4 = 0, \quad B_1 + B_2 + B_3 + B_4 = 0.$$

Similarly for the circle (d) we have

$$A_1 + A_2 + B_3 + C_4 = 0, \text{ which determines } C_4;$$

and for the circle (e) we have

$$B_1 + B_2 + C_3 + A_4 = 0, \text{ which determines } C_3.$$

Again, if C_1 is given, we have from (f)

$$C_1 + C_2 + A_3 + B_4 = 0, \text{ which determines } C_2.$$

Again, adding the last three identities, we have

$$(C_1 + C_2 + C_3 + C_4) + (A_1 + A_2 + A_3 + A_4) + (B_1 + B_2 + B_3 + B_4) = 0;$$

therefore $\qquad C_1 + C_2 + C_3 + C_4 = 0;$

therefore the points C_1, C_2, C_3, C_4 lie on a circle (c).

The other part of the proposition is not true.

4723. (By Prof. Crofton.)—Give a geometrical method of drawing normals to a parabola from a point on the curve.

I. *Solution by* Professor Minchin.

When three normals meet in a point, the chord joining the feet of two makes the same angle with the axis as that made by the line joining the vertex to the foot of the third. Now it is easy to show that, when the point of meeting is on the curve, the chord joining the feet of the two normals passes through a fixed point on the axis at a distance from the vertex equal to the semi-latus-rectum. Hence the construction is,—join the given point on the curve to the vertex, and from the point on the axis distant from the vertex by the length of the semi-latus-rectum draw a line making with the axis the same angle as that made by the line first drawn. We then get the feet of the two required normals.

II. *Solution by* R. F. Davis, Nilkanta Sarkar, *and others.*

The following property will be of use. If from any point T on the straight line $x = 2a$ tangents TP, TQ be drawn to the parabola $y^2 = 4ax$, the normals at P, Q will intersect in a point O on the curve. [For, adopting the m-variable, the points on the curve the normals at which pass

through any point (X, Y) are given by the equation $m^3Y+m^2(X-2a)-a=0$. (Wolstenholme's *Problems*.)] If X, Y be on the curve, put

$$X=\frac{a}{\mu^2},\quad Y=\frac{2a}{\mu}.$$

Then, rejecting the root μ, we have

$$m^2+\frac{m}{2\mu}+\frac{1}{2}=0.$$

Let m_1, m_2 be the roots of this equation;

then $\frac{a}{m_1m_2}$, $a\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$,—the coordinates of the pole of the line joining m_1m_2,—become $2a$ and $-\frac{a}{m}$. Moreover, the sum of the ordinates of O, P, Q is zero. (Todhunter's *Conics*.) Hence ordinate of T = semi-sum of ordinates of P, Q = ½ the ordinate of O; T and O being on opposite sides of the axis. Thus, if O is given, T is known, and therefore P and Q.

4702. (By C. W. Merrifield.)—Two right cones have their vertices in a plane normal to their axes; prove that (1) the projection, on that plane, of their intersection, is a circle relatively to which the projections of the vertices are inverse points; and (2) the intersection is also that of a sphere with a parabolic cylinder.

I. *Solution by* J. J. Walker.

1. The two cones may be represented by the equations

$$x^2-k\left\{(y+a)^2+z^2\right\}=0,\quad x^2-k'\left\{(y-a)^2+z^2\right\}=0,$$

when the projection of their curve of intersection on the plane to which their axes are normal will be

$$(k-k')(y^2+z^2)+2(k+k')ay+(k-k')a^2=0,$$

a circle the distance of the centre of which from origin is equal to $\frac{k'+k}{k'-k}a$, and the square of its radius to $\frac{4kk'a^2}{(k'-k)^2}$. The distances of the vertices of the cones from origin are a, $-a$ respectively, and therefore from the centre of the above circle $\frac{2ka}{k-k'}$ and $\frac{2k'a}{k-k'}$; the product of which distances is equal to the square of the radius.

2. Multiplying the second equation above by λ, and adding it to the

first, the result will be satisfied by any point common to the two cones; but this result may be thrown into the form

$$\left\{1+k+\lambda(1+k')\right\}x^2-2(k-\lambda k')\,a\left(y+\frac{k+\lambda k'}{k-\lambda k'}a\right)$$
$$-(k+\lambda k')(x^2+y^2+z^2-a^2)=0,$$

which form admits of an obvious generalization.

II. *Solution by* F. D. Thomson.

Take the line joining the vertices as the axis of x, and the plane containing the axes as the plane of xz. Then the equations to the first and second cones (vertex origin) are of the form

$$z^2=(x^2+y^2+z^2)\cos^2\alpha,\quad\text{or}\quad x^2+y^2=z^2\tan^2\alpha\ \ldots\ldots\ldots\ldots\ldots(1),$$

$$z^2=\left\{(x-a)^2+y^2+z^2\right\}\cos^2\beta,\quad\text{or}\quad (x-a)^2+y^2=z^2\tan^2\beta\ \ldots\ldots(2).$$

The projection on the plane of xy of their curve of intersection is

$$(x^2+y^2)\tan^2\beta=\left\{(x-a)^2+y^2\right\}\tan^2\alpha,$$

a circle which meets the axis of x in the points given by

$$\pm\,x\tan\beta=(x-a)\tan\alpha.$$

Let x_1, x_2 be the two values of x; then we have

$$x_1=\frac{a\tan\alpha}{\tan\alpha-\tan\beta},\quad x_2=\frac{a\tan\alpha}{\tan\alpha+\tan\beta},\quad\text{therefore}\quad \frac{1}{x_1}+\frac{1}{x_2}\ \frac{2}{a},$$

therefore each vertex lies on the polar of the other with respect to the circle.

Again, we have

$$x^2+y^2+z^2(1-\sec^2\alpha)=0\quad\text{and}\quad (x-a)^2+y^2+z^2(1-\sec^2\beta)=0\ldots(3,4),$$

therefore (3) $\times\sec^2\beta-(4)\times\sec^2\alpha$ gives

$$(x^2+y^2+z^2)(\sec^2\beta-\sec^2\alpha)+(2ax-a^2)\sec^2\alpha=0,$$

which is a sphere through their intersection.

Again, subtracting (2) from (3), we obtain

$$2ax-a^2+z^2(\sec^2\beta-\sec^2\alpha)=0,$$

which is a parabolic cylinder through their intersection.

4647. (By R. W. Genese.)—If $u=r^{-1}=f(\theta)$ be the polar equation to any curve; SY, SZ perpendiculars from the origin S on the tangent and normal at the point P $(\theta=\alpha)$; prove that (1) the equation to YZ will

be $\qquad f'(\alpha)\, u = 2f'(\alpha) \cos \theta - \alpha + \left\{[f'(\alpha)]^2 - [f(\alpha)]^2\right\} \sin(\theta-\alpha);$

(2) if PN be perpendicular to $\theta = 0$, the equation to NY is

$$\cos \alpha \,.\, u = f(\alpha) \cos \theta + f'(\alpha) \sin \theta;$$

(3) the equation to the locus of Y is

$$f(\alpha)\, u = \left\{[f(\alpha)]^2 - [f'(\alpha)]^2\right\} \cos(\theta-\alpha) + 2f(\alpha) f'(\alpha) \sin(\theta-\alpha).$$

I. *Solution by* R. F. Davis.

Let $SY = p$, $SZ = q$, and $SP = r$. Assume $u = A \cos(\theta-\alpha) + B \sin(\theta-\alpha)$ as the equation to YZ. At the points Y, Z, we have

$$u = \frac{1}{p} \text{ and } \theta = \alpha - \tan^{-1}\frac{q}{p},$$

$$u = \frac{1}{q} \text{ and } \theta = \alpha + \tan^{-1}\frac{p}{q}.$$

These values of the variables must satisfy the assumed equation; whence

$$A = \frac{2}{r}, \text{ and } B = \frac{p^2-q^2}{pqr};$$

$$\frac{p}{q} = \tan SPY = -\frac{f(\alpha)}{f'(\alpha)}.$$

Substituting in the values of A, B, we get the required form. Next, the polar equation to NY may be found by transforming the Cartesian equation, which is obviously

$$y = \frac{p}{q}(x - r \cos \alpha).$$

Lastly, assume $u = A \cos(\theta-\alpha) + B \sin(\theta-\alpha)$ as the equation to the tangent at Y to the locus of Y. Draw SL perpendicular to this tangent; then it is known that $SL = \frac{p^2}{r}$ and that the angles SYL, SPY are equal. At the point Y, as before, $u = \frac{1}{p}$ and $\theta = \alpha - \tan^{-1}\frac{q}{p}$; and at the point L, $u = \frac{r}{p^2}$ and $\theta = \alpha - 2\tan^{-1}\frac{q}{p}$. These values of the variables must satisfy the assumed equation; whence

$$A = \frac{p^2-q^2}{p^2 r}, \text{ and } B = -\frac{2q}{r}.$$

II. *Solution by* C. Leudesdorf.

If SY make an angle θ with the initial line, and if f denote $f(\alpha)$ and f' denote $f'(\alpha)$, we shall have

$$\theta = \alpha - \tfrac{1}{2}\pi + \tan^{-1}\left(-u\frac{d\theta}{du}\right) = \alpha - \tfrac{1}{2}\pi - \tan^{-1}\frac{f}{f'},$$

therefore $\qquad \dfrac{\cos(\theta-\alpha)}{f} = \dfrac{\sin(\theta-\alpha)}{f'} = \dfrac{1}{(f^2+f'^2)^{\frac{1}{2}}} + SY,$

since $$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2;$$

hence we have $$(f^2 - f'^2)\cos(\theta - \alpha) + 2ff'\sin(\theta - \alpha) = (f^2 + f'^2)\frac{f}{u} = fu$$

as the equation of the locus of Y.

Again $$\frac{\cos\theta}{f\cos\alpha - f'\sin\alpha} = \frac{\sin\theta}{f\sin\alpha + f'\cos\alpha} + \frac{1}{u} \quad \text{.............. (1).}$$

The equation of NY, since it passes through N, is of the form

$$\frac{\cos\alpha}{f}u = \cos\theta + \lambda\sin\theta \quad \text{........................... (2).}$$

And the equation of YZ, since it passes through the middle point of SP, is of the form

$$u(\cos\alpha + \mu\sin\alpha) = 3f(\cos\theta + \lambda\sin\theta) \text{.....................(3).}$$

Substituting in (2) and (3) from (1), we find

$$\lambda = \frac{f'}{f}, \text{ and } \mu = \frac{(f'^2 - f^2)\cos\alpha + 2ff'\sin\alpha}{-(f^2 - f'^2)\sin\alpha + 2ff'\cos\alpha},$$

whence the equations to NY and YZ are found to be the same as those given in the Question.

4747. (By W. A. Whitworth.)—A shot is fired in an atmosphere in which the resistance varies as the cube of the velocity. If f be the retardation when the shot is ascending at an inclination α to the horizon, f_0 when it is moving horizontally, and f' when it is descending at an inclination α to the horizon; then

$$\frac{1}{f'} + \frac{1}{f} = \frac{2\cos^3\alpha}{f_0}, \text{ and } \frac{1}{f'} - \frac{1}{f} = \frac{2\sin\alpha(3 - 2\sin^2\alpha)}{g}.$$

Solution by R. F. Davis.

Let v be the velocity, kv^3 the retardation, and ϕ the angle the direction of motion makes with the horizontal at the time t. Then we have

$$\frac{d}{dt}(v\cos\phi) = -kv^3\cos\phi; \text{ and } \frac{d}{dt}(v\sin\phi) = -g - kv^3\sin\phi.$$

Hence $$d(v\cos\phi) : d(v\sin\phi) = kv^3\cos\phi : g + kv^3\sin\phi.$$

It will be found that this proportion reduces to $g \,.\, d(v\cos\phi) = kv^4 \,.\, d\phi$;

and so $$\frac{d(v\cos\phi)}{kv^4\cos^4\phi} = \frac{d\phi}{g\cos^4\phi}.$$

Integrating, we get $$\frac{1}{kv_0^3} - \frac{1}{kv^3\cos^3\phi} = \frac{3}{g}\left(\tan\phi + \frac{\tan^3\phi}{3}\right),$$

if v_0 be the velocity when the motion is horizontal. Thus

$$\frac{\cos^3\phi}{kv_0^3} - \frac{1}{kv^3} = \frac{\sin\phi\,(3-2\sin^2\phi)}{g}.$$

Then put $\phi = \alpha$ and $-\alpha$ successively, and the required results follow.

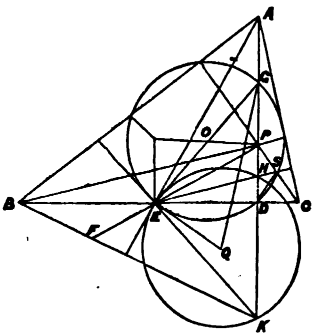
4720. (By R. Tucker.)—AE, AD are the median line and perpendicular respectively from A to the base BC of the triangle ABC, and P, H, K are the ortho-centres of the triangles ABC, AEC, ABE; show that (1) P, H, D, K form a harmonic range; (2) the nine-point circle of ABC and the circle EHK cut each other orthogonally.

I. *Solution by* R. W. Genese; F. D. Thomson; *and others.*

1. EK is parallel to PC, and therefore bisects BP in M say; but EH is parallel to BP; therefore E (BMPH) is an harmonic pencil, and it meets AD in the range DKPH.

2. If F, G be the middle points of CA and AB, then EF and EG are parallel to AB and AC, and therefore perpendicular to EK and EH, and GF is perpendicular to HK; thus the triangle EFG is similar to EKH turned through a right angle. Therefore the circles through EFG and EKH will have their tangents at E at right angles, and cut orthogonally.

II. *Solution by* S. A. Renshaw, Nilkanta Sarkar, *and others.*



1. Construct the figure, and let O and Q be the centres of the nine-point circle of ABC, and of the circle circumscribed about EHK. Produce EH to meet BK in F. Then, since HC, BK are parallel, being both perpendicular to AE, and BE = EC, therefore (Euc. I. 26) EH = EF. But EH is parallel to BP; therefore BP, BH, BD, BK form an harmonic pencil, and P, H, D, K an harmonic range.

2. Produce EH to meet PC in S; then since BE = EC and BP, ES are parallel (being both perpendicular to AC), therefore (Euc. VI. 2) PS = SC; therefore the nine-point circle passes through S. Join DS; then PDC being a right-angled triangle, DS = SP and $\angle$ SPD = SDP. But $\angle$ SPD = PKE, since PC, EK are parallel, and SDP = SEG by the circle; therefore HKE = HEG; therefore GE touches the circle Q. But GE is a diameter of the circle O, therefore the circles cut each other othogonally.

Corollary.—We have $GE^2 = KG \cdot GH$.

4710. (By E. B. ELLIOTT.)—If $\phi(x)$ is any function of x which approaches definite limits as x becomes zero and infinite, prove that

$$\int_0^a \frac{\phi(bx)}{x}\,dx - \int_0^b \frac{\phi(ax)}{x}\,dx = \phi(0)\log\frac{a}{b} \text{ .... (1)},$$

$$\int_a^\infty \frac{\phi(bx)}{x}\,dx - \int_b^\infty \frac{\phi(ax)}{x}\,dx = \phi(\infty)\log\frac{b}{a} \text{ .... (2)}.$$

I. *Solution by* F. D. THOMSON.

1. By Maclaurin's Theorem, under the conditions stated,

we have $$\phi(ax) = \phi(0) + xa\phi'(0) + \frac{x^2a^2}{\lfloor 2}\phi''(0) + \&c.\,;$$

therefore $$\frac{\phi(ax)}{x} = \frac{\phi(0)}{x} + a\phi'(0) + \frac{xa^2}{\lfloor 2}\phi''(0) + \&c.,$$

therefore $$\int \frac{\phi(ax)}{x}\,dx = \log x\,\phi(0) + ax\phi'(0) + \frac{a^2x^2}{2\lfloor 2}\phi''(0) + \&c.$$
$$= \log x\,\phi(0) + \mathrm{F}(ax), \quad \text{suppose.}$$

Similarly $\int \frac{\phi(bx)}{x}\,dx = \log x\,\phi(0) + \mathrm{F}(bx)$; hence we have

$$\int_0^a \frac{\phi(bx)}{x}\,dx - \int_0^b \frac{\phi(ax)}{x}\,dx = \phi(0)\left[\log x\Big\}_0^b - \log x\Big\}_0^a\right] = \phi(0)\log\frac{b}{a}.$$

2. If $x = \dfrac{1}{y}$, we shall have

$$\phi(ax) = \phi\left(\frac{a}{y}\right) = \phi(0) + ya\phi'(0) + \frac{y^2a^2}{\lfloor 2}\phi''(0) + \&c.,$$

where $\phi(0)$ denotes the result of putting $y = 0$ or $x = \infty$ in $\phi\left(\frac{a}{y}\right)$,

and $$\frac{\phi(ax)}{x}\,dx = -y\phi\left(\frac{a}{y}\right)\frac{dy}{y^2} = -\frac{1}{y}\phi\left(\frac{a}{y}\right)dy\,;$$

therefore $$\int_a^\infty \frac{\phi(ax)}{x}\,dx = -\Big[\log y\,\phi(0) + \mathrm{F}(ay), \text{ suppose}\Big]_{\frac{1}{a}}^{0}$$

Similarly $$\int \frac{\phi(bx)}{x}\,dx = -\Big[\log y\,\phi(0) + \mathrm{F}(by)\Big]_{\frac{1}{b}}^{0}\,; \text{ therefore}$$

$$\int_a^\infty \frac{\phi(bx)}{x}\,dx - \int_b^\infty \frac{\phi(ax)}{x}\,dx = \phi(0)\left\{\log\frac{1}{a} - \log\frac{1}{b}\right\} = \phi(\infty)\log\frac{b}{a},$$

replacing $\phi(0)$ by $\phi(\infty)$ to denote the value of $\phi(x)$ when $x = \infty$.

II. *Solution by the* PROPOSER.

1. We have $\int_{a'}^{a} \frac{\phi(bx)-\phi(b'x)}{x}\,dx = \int_{a'}^{a}\int_{b'}^{b} \phi'(xy)\,dy\,dx$

$$= \int_{b'}^{b} \frac{\phi(ay)-\phi(a'y)}{y}\,dy \quad \ldots\ldots\ldots\ldots (A).$$

In this put $a' = b' = 0$; then we have

$$\int_0^a \frac{\phi(bx)-\phi(0)}{x}\,dx = \int_0^b \frac{\phi(ax)-\phi(0)}{x}\,dx;$$

that is, $$\int_0^a \frac{\phi(bx)}{x}\,dx - \int_0^b \frac{\phi(ax)}{x}\,dx = \int_b^a \phi(0)\frac{dx}{x}$$

$$= \phi(0)\log\frac{a}{b} \quad \ldots\ldots\ldots\ldots (B).$$

2. Again in (A) put $a = b = \infty$; then we have

$$\int_{a'}^{\infty} \frac{\phi(\infty)-\phi(b'x)}{x}\,dx = \int_{b'}^{\infty} \frac{\phi(\infty)-\phi(a'x)}{x}\,dx;$$

that is, $\int_{a'}^{\infty} \frac{\phi(b'x)}{x}\,dx - \int_{b'}^{\infty} \frac{\phi(a'x)}{x}\,dx = \int_{a'}^{b'} \phi(\infty)\frac{dx}{x} = \phi(\infty)\log\frac{b'}{a'}$ ... (C).

It may be remarked that the identity (A) affords an immediate proof of Frullani's theorem. In it put $a' = 0$, $a = \infty$; then we have

$$\int_0^{\infty} \frac{\phi(bx)-\phi(b'x)}{x}\,dx = \int_{b'}^{b} \frac{\phi(\infty)-\phi(0)}{x}\,dx = \left\{\phi(\infty)-\phi(0)\right\}\log\frac{b}{b'},$$

which becomes Frullani's theorem in the case when $\phi(\infty) = 0$.

4717. (By J. J. WALKER.)—In elliptic motion, if ω, ω' are the angular velocities about the focus which is the centre of force and the second focus respectively, and ϖ the value of ω for the end of the latus-rectum; prove that

$$\frac{1-e^2}{\omega'} - \frac{e^2\sin^2\theta}{\omega} = \frac{1}{\varpi}.$$

Solution by F. D. THOMSON, R. F. DAVIS, *and others.*

With the usual notation, we have $r^2\frac{d\theta}{dt} = h$, so that $\omega = \frac{h}{r^2}$. Also, since the tangent makes equal angles with the focal distances, $\omega r = \omega' r'$;

therefore $\varpi' = \frac{h}{rr'}$. But $rr' = a^2 - e^2x^2 = a^2(1-e^2) + \frac{e^2}{1-e^2}y^2$; hence $(1-e^2)rr' - e^2r^2\sin^2\theta = l^2$, where l is the semi-latus-rectum; or

$$\frac{1-e^2}{\omega'} - \frac{e^2\sin^2\theta}{\omega} = \frac{l^2}{h} = \frac{1}{\varpi}.$$

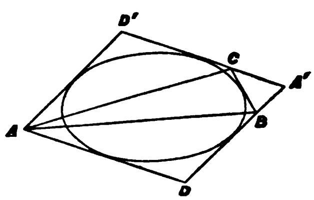
4674. (By C. TAYLOR.)—The sides of a triangle ABC touch a circle and intersect a fourth tangent. If the diameters parallel to the polars of the points of section meet the fourth tangent in a, b, c, prove that Aa, Bb, Cc, are parallel.

Solution by R. F. DAVIS, E. RUTTER, *and many others.*

From the fundamental property that the orthocentre of any triangle circumscribing a parabola lies on the directrix, we may derive, by reciprocation with respect to the focus S, a second property,—that, if S be any point on the circle circumscribing a triangle ABC, and straight lines be drawn through S at right angles to SA, SB, SC meeting the opposite sides in $\alpha\beta\gamma$, $\alpha\beta\gamma$ all lie on a diameter of the circle. Reciprocating again with respect to the centre of the circle, we get the required property.

3861. (By R. W. GENESE.)—A conic is inscribed in a parallelogram; and any tangent to the conic meets the sides opposite to an angle A in B, C. Prove that the triangle ABC is constant in area.

Solution by the PROPOSER.



Let ADA′D′ be the parallelogram. If B be taken on A′D, only one tangent BC can be drawn; therefore C is uniquely determined: similarly C would determine B. If, then, A′B $= x$ and A′C $= y$, the relation between x and y must be of the form

$$axy + bx + cy + d = 0 \text{ ...... (1).}$$

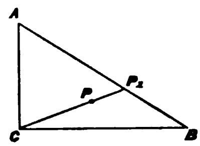
Now, when B is at D, C is at infinity; and, from (1), when $y = \infty$, $ax + c = 0$; therefore $c = -a\,.\,\text{A'D}$. Similarly we may show that $b = -a\,.\,\text{A'D'}$. Thus (1) may be re-written

$$a\,(x - \text{A'D})\,(y - \text{A'D'}) = \text{constant}; \text{ i.e., DB . D'C = constant.}$$

And $\quad$ Area ABC $= \frac{1}{2}$ (AD . AD′ − BD . CD′) sin A = constant.

4427. (By A. MARTIN.)—If M be the mean distance of all points in the surface of a right-angled triangle from the right angle, and M_1 the mean distance of all points in the hypotenuse from the right angle, prove that $M = \frac{2}{3} M_1$.

Solution by REV. J. R. WILSON, A. M. NASH, *the* PROPOSER, *and others.*



1. In the first case, let P be any point in the triangle ABC, $CP = r$, $PCB = \theta$. Then, if n be the number of elemental areas, $nr\,\delta r\,\delta\theta = \frac{1}{2}ab$. Therefore

$$M = \frac{\Sigma(r)}{n} = \frac{2}{ab}\int_0^{\frac{1}{2}\pi}\int_0^{CP_1} r^2\,d\theta\,dr$$

$$= \frac{2a^2\sin^3 B}{3b}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sin^3(\theta+B)}.$$

2. In the second case, let $BP_1 = x$, and let AB be divided into n equal parts δx, so that $n\delta x = c$. Therefore

$$M_1 = \frac{\Sigma(CP_1)}{n} = \frac{1}{c}\int_0^c CP_1\,dx = \frac{\sin B}{c}\int_0^c \frac{x}{\sin\theta}\,dx$$

$$= \frac{a\sin B}{c}\int_0^{\frac{1}{2}\pi}\frac{dx}{d\theta}\cdot\frac{d\theta}{\sin(\theta+B)} = \frac{a^2\sin^2 B}{c}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sin^3(\theta+B)}.$$

Hence, remembering that $b = c\sin B$, we have the result $M = \frac{2}{3}M_1$.

4658. (By Dr. HART.)—Find n numbers, such that, if the cube of their sum be added to each, the sums shall be cubes.

Solution by SAMUEL BILLS.

Let $x_1, x_2, x_3 \ldots\ldots x_n$ denote the required numbers, and s their sum.

Let $\quad s^3 + x_1 = a_1^3 s^3,\; s^3 + x_2 = a_2^3 s^3,\; s^3 + x_3 = a_3^3 s^2,\; \ldots\ldots\; s^3 + x_n = a_n^3 s^3;$

then $\quad x_1 = (a_1^3 - 1)s^3,\; x_2 = (a_2^3 - 1)s^3,\; \ldots\ldots\; x_n = (a_n^3 - 1)s^3 \;\ldots\ldots\ldots(A).$

By addition, $s = (a_1^3 + a_2^3 + a_3^3 \ldots\ldots - n)s^3$, or $\dfrac{1}{s^2} = a_1^3 + a_2^3 + a_3^3 \ldots\ldots - n.$

Put $\qquad a_3^3 + a_4^3 + a_5^3 + \ldots\ldots a_n^3 - n = S$, and $\dfrac{1}{s^2} = K^2;$

therefore $\qquad K^2 = a_1^3 + a_2^3 + S.$

Now put $a_1 = \frac{3}{2} + v$, and $a_2 = \frac{3}{2} - v$; therefore

$$K^2 = \tfrac{27}{4} + 9v^2 + S, \text{ or } K^2 - 9v^2 = \tfrac{27}{4} + S = fg \text{ (suppose).}$$

Assume $K+3v=f$, and $K-3v=g$; therefore

$$K=\tfrac{1}{2}(f+g) \text{ and } v=\tfrac{1}{6}(f-g).$$

This furnishes a complete solution to the question. It is evident, from (A), that $a_1, a_2, a_3 \ldots\ldots$ must all be greater than unity; also v, or $\tfrac{1}{6}(f-g)$, must be less than $\tfrac{1}{2}$, or $f-g$ must be less than 3.

To give an example in numbers: suppose $n=6$, and take $a_3=\tfrac{3}{2}$, $a_4=2$, $a_5=\tfrac{5}{2}$, $a_6=3$; then will $S=48$, which gives $K^2-9v^2=\tfrac{219}{4}=fg$. Now take $f=\tfrac{15}{2}$, and $g=\tfrac{73}{10}$, therefore we find $K=\tfrac{37}{5}$ and $v=\tfrac{1}{30}$; whence $a_1=\tfrac{23}{15}$, $a_2=\tfrac{22}{15}$; also $s=\tfrac{5}{37}$; and the six numbers will be

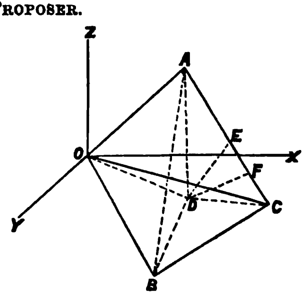
$$x_1=\left\{(\tfrac{23}{15})^3-1\right\}(\tfrac{5}{37})^3,\quad x_2=\left\{(\tfrac{22}{15})^3-1\right\}(\tfrac{5}{37})^3,\quad x_3=\left\{(\tfrac{3}{2})^3-1\right\}(\tfrac{5}{37})^3,$$

$$x_4=\left\{(2)^3-1\right\}(\tfrac{5}{37})^3,\quad x_5=\left\{(\tfrac{5}{2})^3-1\right\}(\tfrac{5}{37})^3,\quad x_6=\left\{(3)^3-1\right\}(\tfrac{5}{37})^3.$$

These numbers satisfy the conditions of the question; and any number of other answers may be found at pleasure.

4541. (By S. Tebay.)—Shew that (1) the average area of the shadow of a regular tetrahedron of edge a on a plane perpendicular to the direction of a distant luminary is $\cdot 40042a^2$; (2) the probability that the shadow will be a triangle is $\pi^{-2}(6\cos^{-1}\tfrac{1}{3}-2\pi)$.

Solution by the Proposer.



Let OABC be the tetrahedron, O the origin, OD perpendicular to ABC, which we may suppose to be in the plane yz, E the middle of AC, and DF the intersection of xz with ABC. Let (x, y, z), (x', y', z'), (x'', y'', z'') be the coordinates of A, B, C; angle DOX $=\theta$, EDF $=\phi$, and AB $=1$. Then we have

$$OD=\sqrt{\left(\frac{2}{3}\right)},\quad AD=\frac{1}{\sqrt{3}},$$

$$y=\frac{1}{\sqrt{3}}\sin\left(\phi+\frac{\pi}{3}\right),\qquad z=\frac{1}{\sqrt{3}}\cos\theta\cos\left(\phi+\frac{\pi}{3}\right)-\sqrt{\tfrac{2}{3}}\sin\theta,$$

$$y'=-\frac{1}{\sqrt{3}}\sin\phi,\qquad z'=-\frac{1}{\sqrt{3}}\cos\theta\cos\phi-\sqrt{\tfrac{2}{3}}\sin\theta,$$

$$y''=-\frac{1}{\sqrt{3}}\sin(\phi+\tfrac{2}{3}\pi),\qquad z''=-\frac{1}{\sqrt{3}}\cos\theta\cos(\phi+\tfrac{2}{3}\pi)-\sqrt{\tfrac{2}{3}}\sin\theta.$$

When the shadow is a triangle, we may consider the projection of ABC

on the plane yz. The area of ABC is $\frac{1}{4}\sqrt{3}$, and the projection on yz is $\frac{1}{4}\sqrt{3}\cos\theta$; the limits of θ are 0 and $\sin^{-1}\dfrac{1}{\sqrt{(9-8\sin^2\phi)}}$, and of ϕ, 0 and $\frac{1}{3}\pi$.

This gives
$$\frac{3}{4\sqrt{3}}\int_0^{\frac{1}{3}\pi}\frac{d\phi}{\sqrt{(9-8\sin^2\phi)}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

When the shadow is a quadrilateral, we may consider the projection of ABC + AOC on yz. The projection of AOC $=\frac{1}{2}(yz''-y''z)=\dfrac{1}{4\sqrt{3}}\cos\theta-\dfrac{1}{\sqrt{6}}\sin\theta\cos\phi$; and the sum $=\dfrac{1}{\sqrt{3}}\cos\theta-\dfrac{1}{\sqrt{6}}\sin\theta\cos\phi$. The integral will consist of two parts: in the first case, the limits of θ are $\sin^{-1}\dfrac{1}{\sqrt{(9-8\sin^2\phi)}}$ and $\sin^{-1}\dfrac{1}{\sqrt{3}}$, and of ϕ, 0 and $\dfrac{\pi}{3}$. This gives
$$\frac{\pi}{9}+\frac{1}{2\sqrt{3}}-\frac{1}{4\sqrt{3}}\int_0^{\frac{1}{3}\pi}\sqrt{(9-8\sin^2\phi)}\,d\phi-\frac{3}{4\sqrt{3}}\int_0^{\frac{1}{3}\pi}\frac{d\phi}{\sqrt{(9-8\sin^2\phi)}}\ldots\text{(B)}.$$

To find the limits in the second case, make the face BOC perpendicular to yz, that is, put the area of its projection on $yz=0$, or $y''z'-y'z''=0$; and we find $\cos(\phi+\frac{1}{3}\pi)+\dfrac{\cot\theta}{2\sqrt{2}}=0$. Hence the limits of θ are $\sin^{-1}\dfrac{1}{\sqrt{3}}$ and
$$\sin^{-1}\frac{1}{\sqrt{\left\{9-8\sin^2(\phi+\frac{1}{3}\pi)\right\}}},\ \text{and of } \phi,\ \tfrac{1}{3}\pi \text{ and } \tfrac{1}{6}\pi.$$

The remaining portion of the integral is therefore
$$\frac{\pi}{18}-\frac{7}{24}+\frac{7}{8\sqrt{3}}+\frac{1}{8\sqrt{3}}\int_{\frac{1}{3}\pi}^{\frac{1}{6}\pi}\sqrt{\left\{9-8\sin^2(\phi+\tfrac{1}{3}\pi)\right\}}\,d\phi$$
$$+\frac{7}{8\sqrt{3}}\int_{\frac{1}{3}\pi}^{\frac{1}{6}\pi}\frac{d\phi}{\sqrt{\left\{9-8\sin^2(\phi+\frac{1}{3}\pi)\right\}}}$$
$$=\frac{\pi}{18}-\frac{7}{24}+\frac{7}{8\sqrt{3}}+\frac{1}{8\sqrt{3}}\int_{\frac{2}{3}\pi}^{\frac{1}{2}\pi}\sqrt{(9-8\sin^2\phi)}\,d\phi+\frac{7}{8\sqrt{3}}\int_{\frac{2}{3}\pi}^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(9-8\sin^2\phi)}}$$
$$\ldots\ldots\ldots\ldots\text{(C)}$$

writing ϕ for $\phi+\frac{1}{3}\pi$. Therefore
$$\text{B}+\text{C}=\frac{\pi}{6}-\frac{7}{24}+\frac{11}{8\sqrt{3}}-\frac{3}{4\sqrt{3}}\text{E}\left(\frac{2\sqrt{2}}{3},\frac{\pi}{3}\right)+\frac{3}{8\sqrt{3}}\text{E}\left(\frac{2\sqrt{2}}{3},\frac{\pi}{2}\right)$$
$$-\frac{3}{8\sqrt{3}}\text{E}\left(\frac{2\sqrt{2}}{3},\frac{2}{3}\pi\right)-\frac{1}{4\sqrt{3}}\text{F}\left(\frac{2\sqrt{2}}{3},\frac{\pi}{3}\right)+\frac{7}{8\sqrt{3}}\text{F}\left(\frac{2\sqrt{2}}{3},\frac{\pi}{2}\right)$$
$$-\frac{7}{8\sqrt{3}}\text{F}\left(\frac{2\sqrt{2}}{3},\frac{2}{3}\right)\pi\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(D)}.$$

Hence the sum of all the shadows is

$$48A + 36D = 6\pi - 10\tfrac{1}{2}$$

$$+ \frac{99}{2\sqrt{3}} - \frac{27}{\sqrt{3}}\,E\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{3}\right) + \frac{27}{2\sqrt{3}}\,E\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{2}\right) - \frac{27}{2\sqrt{3}}\,E\left(\frac{2\sqrt{2}}{3}, \frac{3}{3}\pi\right)$$

$$+ \frac{3}{\sqrt{3}}\,F\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{3}\right) + \frac{21}{2\sqrt{3}}\,F\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{2}\right) - \frac{21}{2\sqrt{3}}\,F\left(\frac{2\sqrt{2}}{3}, \frac{2}{3}\pi\right).$$

By the theory of definite integrals, we have

$$E\left(\frac{2\sqrt{2}}{3}, \frac{2}{3}\pi\right) = 2\,E\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{2}\right) - E\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{3}\right),$$

$$F\left(\frac{2\sqrt{2}}{3}, \frac{2}{3}\pi\right) = 2\,F\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{2}\right) - F\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{3}\right).$$

And, by the theory of elliptic functions, we have

$$E\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{3}\right) = \tfrac{2}{3}E\left(\frac{1}{2}, \frac{\pi}{2}\right) - \tfrac{1}{4}F\left(\frac{1}{2}, \frac{\pi}{2}\right) + \frac{1}{3}$$

$$F\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{3}\right) = \tfrac{3}{4}F\left(\frac{1}{2}, \frac{\pi}{2}\right),\quad F\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{2}\right) = \tfrac{3}{2}F\left(\frac{1}{2}, \frac{\pi}{2}\right).$$

Making these substitutions, the sum of the shadows is

$$6\pi - 10\tfrac{1}{2} + \frac{45}{\sqrt{3}} - \frac{9}{\sqrt{3}}\,E\left(\frac{1}{2}, \frac{\pi}{2}\right) - \frac{27}{2\sqrt{3}}\,E\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{2}\right) - \frac{9}{4\sqrt{3}}\,F\left(\frac{1}{2}, \frac{\pi}{2}\right).$$

The number of shadows $= 4\pi^2$; therefore the average area is

$$\frac{1}{4\pi^2}\left\{6\pi - 10\tfrac{1}{2} + \frac{45}{\sqrt{3}} - \frac{9}{\sqrt{3}}\,E\left(\frac{1}{2}, \frac{\pi}{2}\right) - \frac{27}{2\sqrt{3}}\,E\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{2}\right) - \frac{9}{4\sqrt{3}}\,F\left(\frac{1}{2}, \frac{\pi}{2}\right)\right\}.$$

If a be the edge of the tetrahedron, the average area is $\cdot 40042a^2$.

The number of triangles is equal to 8 times the solid angle of the tetrahedron, that is, equal to $8\left(3\cos^{-1}\tfrac{1}{3} - \pi\right)$; and the probability that the shadow is a triangle is therefore $\dfrac{6\cos^{-1}\frac{1}{3} - 2\pi}{\pi^2}$.

3891. (By Dr. HART.)—To find n numbers whose sum is a square, and the sum of their squares a biquadrate.

I. *Solution by the* PROPOSER.

1. Let px^2-ax, px^2+ax; qx^2-bx, qx^2+bx; rx^2-cx, rx^2+cx; sx^2-dx, sx^2+dx; ...... Nx^2-Zx, Nx^2+Zx; and Sx^2, be the numbers, the last being used when n is odd. Then their sum divided by x^2 gives

$$2(p+q+r+s+\ \ldots\ldots\ +N)+S=\square\ \ldots\ldots\ldots\ldots\ (1),$$

and the sum of their squares is

$$\left\{2(p^2+q^2+r^2+s^2+\ldots\ldots+N^2)+S^2\right\}x^4+2(a^2+b^2+c^2+d^2+\ldots\ldots+Z^2)x^2$$

whence we have $\qquad = \text{biquadrate} = \dfrac{m^4}{n^4}x^4,$

$$x^2=\frac{2n^4(a^2+b^2+c^2+d^2+\ldots\ldots+Z^2)}{m^4-\left\{2(p^2+q^2+r^2+s^2+\ldots\ldots+N^2)+S^2\right\}n^4}.$$

2. EXAMPLES.—For *four* numbers, take $p=1$, $q=1$, then $(1)=4=\square$. Let $m=3$, $n=2$, then $x^2=\frac{32}{17}(a^2+b^2)=\frac{64}{34}(a^2+b^2)$. Let $a=5$, $b=3$, then $x^2=64$, $x=8$, whence, after dividing by 4, the four numbers are 6, 10, 22, 26. Also, by another process, I find 2, 22, 26, 94.

3. For *five* numbers take $p=1$, $q=1$, $S=5$, then $(1)=9=\square$. Let $m=7$, $n=3$, then $x^2=\frac{162}{52}(a^2+b^2)=\frac{81}{26}(a^2+b^2)$. Let $a=5$, $b=1$; then $x^2=81$, $x=9$, whence, dividing by 9, the numbers are 4, 8, 10, 14, 45.

4. For *six* numbers, take 3, 3, 2 for p, q, r, respectively; then $(1)=16$. Let $m=8$, $n=3$, then $x^2=\frac{162}{532}(a^2+b^2+c^2)=\frac{81}{266}(a^2+b^2+c^2)$. Let $a=11$, $b=9$, $c=8$, then $x^2=81$, $x=9$; and dividing by 9 the nos. are 10, 16, 18, 26, 36, 38. Or, if $a=12$, $b=11$, $c=1$, then the nos. will be 15, 16, 17, 19, 38, 39; if $a=1$, $b=11$, $c=12$, then we have 6, 16, 26, 28, 30, 38. Again, let $p=2$, $q=2$, $r=4$, then $(1)=16$. Let $m=3$, $n=1$, then $x^2=\frac{2}{33}(a^2+b^2+c^2)=\frac{4}{66}(a^2+b^2+c^2)$. Let $a=1$, $b=4$, $c=7$, then $x^2=4$, $x=2$, and we have 0, 2, 6, 10, 16, 30, being the least *five* numbers that satisfy the conditions.

5. For *seven* nos. take $p=4$, $q=5$, $r=6$, $S=6$; then $(1)=36$. Let $m=4$, $n=1$, then $x^2=\frac{2}{66}(a^2+b^2+c^2)=\frac{1}{33}(a^2+b^2+c^2)$. Let $a=1$, $b=4$, $c=4$, then $x^2=1$, $x=1$, and the numbers will be 1, 2, 3, 5, 6, 9, 10. The process is the same for any such numbers.

6. To find *three* numbers, whose sum is a square, and the sum of their squares a biquadrate, I proceed in a somewhat different manner, as follows:—Let $2mp$, $2np$, and $m^2+n^2-p^2$, be the numbers. Their sum is $m^2+n^2-p^2+2mp+2np=\square$, and the sum of their squares is

$$(m^2+n^2+p^2)^2=\text{biquadrate, therefore } m^2+n^2+p^2=\square.$$

Let $\qquad m^2+n^2-p^2+2mp+2np=(m-n+p)^2,$

whence we get $\qquad m=\dfrac{p^2-2np}{n},$

and therefrom we obtain, by substitution,

$$m^2+n^2+p^2=\frac{(p^2-2np)^2}{n^2}+n^2+p^2=\square;$$

whence, multiplying by n^2, $p^4-4np^3+5n^2p^2+n^4=\square$, which put $=(p^2-2np+n^2)^2$. Reducing, we have $p=4n$, whence $m=8n$; and therefore $64n^2$, $8n^2$, $49n^2$ are *three* such numbers. Let $n=1$, then we have 8, 49, 64.

[A solution of this question, by Mr. Bills, may be seen on p. 104 of Vol. XVIII. of the *Reprint*. By the above processes, however, Dr. Hart finds smaller numbers than have hitherto been obtained. For instance, the *triad* of least numbers have usually been thought to be 64, 152, 409, —which are, in fact, those found by Mr. Bills,—but Dr. Hart finds for the *triad* the numbers 8, 49, 64.]

II. *Solution by* ARTEMAS MARTIN.

Let $x_1, x_2, x_3, \dots x_n$ be the n numbers; then

$$x_1+x_2+\dots+x_n=\square,\quad x_1^2+x_2^2+\dots+x_n^2 = \text{a biquadrate} \;\dots\dots\; (1,2).$$

Put $x_1=2a_1s,\ x_2=2a_2s,\ \dots,\ x_{n-1}=2a_{n-1}s,\ x_n=a_1^2+a_2^2+\dots+a_{n-1}^2-s^2$, $m=a_2+a_3+\dots+a_{n-1}$; then

$$2a_1s+2ms+a_1^2+a_2^2+a_3^2+\dots+a_{n-1}^2-s^2=\square=A^2 \;\dots\dots\dots\; (3),$$

$$a_1^2+a_2^2+a_3^2+\dots+a_{n-1}^2+s^2=\square=B^2 \;\dots\dots\dots\dots\dots\; (4);$$

whence $\qquad s(2a_1+2m-2s)=A^2-B^2=(A-B)(A+B) \;\dots\dots\dots\dots\; (5).$

Take $s=A-B$, then $A+B=2a_1+2m-2s$, $A=a_1+m-\frac{1}{2}s$.

Substituting this value of A in (3) and reducing, we get

$$a_1=\frac{a_2^2+a_3^2+a_4^2+\dots+a_{n-1}^2-m^2+3ms-\frac{5}{4}s^2}{2m-3s},$$

where $a_2, a_3, a_4, \dots a_{n-1}, s$ may have any values that will give a_1 positive, and $a_1^2+a_2^2+a_3^2+\dots+a_{n-1}^2>s^2$.

Example.—Let $n=4$, then $a_1=\dfrac{3(a_2+a_3)s-2a_2a_3-\frac{5}{4}s^2}{2(a_2+a_3)-3s}$. Take $a_2=8$, $a_3=5$, $s=4$; then $a_1=4$, and the numbers are 32, 40, 64, 89.

[On p. 69 of Vol. XXII. of the *Reprint*, there is a solution by Mr. Evans, which leads to a formula similar to this, but it is not applied to any *numerical* example.]

4751 (By J. J. SYLVESTER, F.R.S.)—The length of a principal axis of a conic of maximum or minimum eccentricity inscribed in a quadrilateral, and the angles which it makes with the sides of the quadrilateral being given, prove that the number of conics, real or imaginary, fulfilling the given conditions, is 16.

Solution by the PROPOSER.

Let $\alpha, \beta, \gamma, \delta$ be the angles which the axes of a quadrilateral make with a fixed line in its plane, and θ the angle which a principal axis of any conic inscribed therein makes with the fixed line. Draw a plane through that axis, making an angle ϕ with the plane of the figure, and project the figure upon the plane so drawn. Call the given axis c and the other axis

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[A solution of this question, by Mr. Bills, may be seen on p. 104 of Vol. XVIII. of the *Reprint*. By the above processes, however, Dr. Hart finds smaller numbers than have hitherto been obtained. For instance, the *triad* of least numbers have usually been thought to be 64, 152, 409, —which are, in fact, those found by Mr. Bills,—but Dr. Hart finds for the *triad* the numbers 8, 49, 64.]

II. *Solution by* ARTEMAS MARTIN.

Let $x_1, x_2, x_3, \ldots x_n$ be the n numbers; then

$$x_1 + x_2 + \ldots + x_n = \square,\quad x_1^2 + x_2^2 + \ldots + x_n^2 = \text{a biquadrate} \ \ldots\ldots (1, 2).$$

Put $x_1 = 2a_1 s$, $x_2 = 2a_2 s$, …, $x_{n-1} = 2a_{n-1}s$, $x_n = a_1^2 + a_2^2 + \ldots + a_n^2$ [illegible], $m = a_2 + a_3 + \ldots + a_{n-1}$; then

$$2a_1 s + 2ms + a_1^2 + a_2^2 + a_3^2 + \ldots + a_{n-1}^2 - \phi = \square = A^2 \ \ldots\ldots (3),$$

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_{n-1}^2 + \phi = \square = B^2 \ \ldots\ldots (4);$$

whence $\quad s(2a_1 + 2m - 2s) = A^2 - B^2 = (A - B)(A + B) \ \ldots\ldots (5).$

Take $s = A - B$, then $A + B = 2a_1 + 2m - 2s$, $A = a_1 + m - \frac{1}{2}s$ [illegible].

Substituting this value of A in (3), and reducing, we get

$$a_1 = \frac{a_2^2 + a_3^2 + a_4^2 + \ldots + a_n^2 \text{ [illegible]}}{2m - 2s},$$

where $a_2, a_3, a_4, \ldots a_{n-1}, s$ may have any values that will give a_1 [illegible], and [illegible].

Example.—Let $s = 4$, then $a_1 =$ [illegible]. [illegible]

[illegible] and the numbers are [illegible].

[illegible]

κ, and make $\cos\phi = \dfrac{\kappa}{c}$; then the projected figure becomes a *circle* inscribed in a quadrilateral the lengths of which are the original sides of the quadrilateral, say a, b, c, d, multiplied respectively by the cosines of the angles which they make with the plane of projection. Hence we have

$$\Sigma \pm a \left\{1-(\sin\phi)^2 \sin(\theta+\alpha)^2\right\}^{\frac{1}{2}} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

and if the conic is one of maximum or minimum eccentricity, $\dfrac{\gamma}{c}$ is a maximum or minimum, and therefore $\delta\phi = 0$. Hence we have a second equation (2), found by differentiating (1) in respect to θ; and now, after differentiation, making $\theta = 0$ in equations (1) and (2), so that $\alpha, \beta, \gamma, \delta$ become the given angles which the axis makes with the sides of the quadrilateral, and writing in the relations between a, b, c, d; $\alpha, \beta, \gamma, \delta$, and eliminating the first four of these quantities, we obtain the determinant

$$\begin{vmatrix} [1-(\sin\alpha)^2(\sin\phi)^2]^{\frac{1}{2}}; & [1-(\sin\beta)^2(\sin\phi)^2]^{\frac{1}{2}}; & [1-(\sin\gamma)^2(\sin\phi)^2]^{\frac{1}{2}}; & [1-(\sin\delta)^2(\sin\phi)^2]^{\frac{1}{2}} \\ \dfrac{\sin\alpha\cos\alpha}{[1-(\sin\alpha)^2(\sin\phi)^2]^{\frac{1}{2}}}; & \dfrac{\sin\beta\cos\beta}{[1-(\sin\beta)^2(\sin\phi)^2]^{\frac{1}{2}}}; & \dfrac{\sin\gamma\cos\gamma}{[1-(\sin\gamma)^2(\sin\phi]^{\frac{1}{2}}}; & \dfrac{\sin\delta\cos\delta}{[1-(\sin\delta)^2(\sin\phi)^2]^{\frac{1}{2}}} \\ \sin\alpha; & \sin\beta; & \sin\gamma; & \sin\delta \\ \cos\alpha; & \cos\beta; & \cos\gamma; & \cos\delta \end{vmatrix}$$

which must be zero when the conditions of the question are fulfilled. Hence, making $\dfrac{1}{(\sin\phi)^2} = \mu$, we find

$$\Sigma \sin(\gamma-\delta)\left[(\sin 2\beta - \sin 2\alpha)\mu + 2(\sin\alpha\sin\beta)\sin(\beta-\alpha)\right]$$
$$.\,[\mu-(\sin\gamma)^2]^{\frac{1}{2}}\,.\,[\mu-(\sin\delta)^2]^{\frac{1}{2}} = 0,$$

where
$$\kappa^2 = \frac{c^2\mu^2}{1-\mu^2}.$$

Taking the Norm of the above expression by giving to each of any three of the four radicals

$$[\mu-(\sin\alpha)^2]^{\frac{1}{2}};\quad [\mu-(\sin\beta)^2]^{\frac{1}{2}};\quad [\mu-(\sin\gamma)^2]^{\frac{1}{2}};\quad [\mu-(\sin\delta)^2]^{\frac{1}{2}},$$

its two algebraical signs, we obtain a rational equation of the degree 8 times 2 in μ, corresponding to the number of solutions of the question, which number is therefore 16, as was to be proved. We may of course express the product of the eight determinants under the form of a single determinant, which equated to zero will represent the above equation in μ.

4770. (By Sir James Cockle.)—Boole propounds (at p. 141 of his *Differential Equations*, 2nd ed.) that a primitive equation $\phi(x, y, c) = 0$ may, by the conversion of c into a function of x, be transformed into any desired equation containing x and y together, or y alone, but not into an equation involving x without y. Test this proposition.

Solution by the PROPOSER.

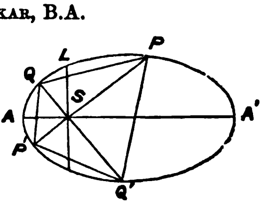
1. From the combination of the desired with the primitive equation, both x and y may disappear, leaving $f(c) = 0$. And $f(c) = 0$ may be self-contradictory or rootless. For instance, the signs in irrational equations may be so fixed that there shall be no root real or unreal.

2. From the primitive $y + x - (y^2 + x^2 + c)^{\frac{1}{2}} = 0$ (which is not porismatic) y can be eliminated by the aid of $c + x^2 = 0$, and we thus obtain $x = 0$. If $xy(y + 2x) = c$, and we take $c + x^3 = 0$, there results $x(y + x)^2 = 0$; whence, not only $y + x = 0$, but also $x = 0$.

3. The proposition is not universally true in either its positive or its negative branch. But the inferences drawn from it by BOOLE (at pp. 143—164, 148—9; see also the *Supplement*, p. 16) may be otherwise supported. Let $V(x, y) = c$ give $y = f(x, c)$; and let $\frac{dy}{dc} = 0$ give $c = \phi(x)$; also let
$$V(x, y) - \phi(x) = \chi(x)\,\psi(x, y),$$
or, more generally, let $\chi(x) = 0$ solve $V(x, y) - \phi(x) = 0$. Then $\chi(x) = 0$ gives $x = a$, a definite constant. Hence $c = \phi(a) =$ a definite constant, say C; and $V(x, y) = C$ is satisfied, independently of y, by $x = a$. But the solution $x = a$, arising from the particular constant value C of c, is not singular, but is a particular primitive. Consequently $\frac{dy}{dc} = 0$ can only lead to singular solutions in which y at least is involved. LACROIX (*Traité Elémentaire*, 5ieme éd., 1837, p. 491) notices that a certain process (which is due to EULER), when applied to $\frac{dy}{dx} - p(x, y)$, leads only to particular (meaning singular) solutions into which y enters.

4750. (By S. WATSON.)—Through the focus of an ellipse two chords are drawn at right angles to each other. Find (1) the average area of the quadrilateral formed by joining their extremities, and (2) the maximum, and (3) the minimum quadrilateral.

Solution by NILKANTA SARKAR, B.A.



The equation to the ellipse in polar coordinates is $l = r(1 - e\cos\theta)$, where $l =$ semi-latus-rectum, and $\theta = \text{A'SP}$;

hence $$\text{PP}' = \frac{2l}{1 - e^2\cos^2\theta},$$

and $$\text{QQ}' = \frac{2l}{1 - e^2\sin^2\theta};$$

therefore the area of the quadrilateral PQP′Q′

$$= \frac{4l^2}{1 - e^2 + e^4\sin^2\theta\cos^2\theta} = \frac{4l^2}{(1 - e^2) + \frac{1}{8}e^4 - \frac{1}{8}e^4\cos 4\theta};$$

hence average area

$$= \int_0^{\pi} \frac{4l^2\, d\theta}{1-e^2+\frac{1}{8}e^4-\frac{1}{8}e^4\cos 4\theta} \div \int_0^{\pi} d\theta$$

$$= 2\int \frac{dz}{1-e^2+(1-e^2+\frac{1}{4}e^4)\,z^2}, \quad \text{if } z = \tan 2\theta$$

$$= \frac{8l^2}{(1-e^2)^2+\frac{1}{4}(1-e^2)\,e^4}.$$

The area is a maximum or a minimum when $(1-e^2)+\frac{1}{8}e^4-\frac{1}{8}e^4\cos 4\theta$ is a minimum or a maximum, which is the case respectively when $\theta = 0$, and $\theta = \frac{1}{4}\pi$. Hence the area is a maximum when the radii vectores coincide with the major axis and the latus rectum, and a minimum for the radii that make half a right angle with the axis.

2942. (By Prof. CLIFFORD, F.R.S.)—If p, q be the foci; P, Q the asymptotes of a conic; θ the angle it subtends at a point a; and [A] the chord it cuts off from a line A: prove that (1) if a line B is drawn through the point a meeting the conic in l, m,

$$al \,.\, am \,.\, \sin \mathrm{BP} \sin \mathrm{BQ} = \frac{ap^2 \,.\, aq^2 \,.\, \sin^2\theta}{pq^2};$$

(2) if from a point b on the line A tangents L, M are drawn to the conic,

$$\sin \mathrm{AL} \sin \mathrm{AM} \,.\, bp \,.\, bq = \frac{\sin^2 \mathrm{AP} \sin^2 \mathrm{AQ} \,.\, [\mathrm{A}]^2}{\sin^2 \mathrm{PQ}},$$

where al means the distance between the points a, l, and BP means the angle between the lines B, P; also (3) find analogous propositions for a curve of any order on a plane or on a sphere.

Solution by the Rev. F. D. THOMSON, M.A.

1. The equation to a curve of order m is of the form

$$\mathrm{S} \equiv (\lambda_1 x + \mu_1 y + \nu_1)(\lambda_2 x + \mu_2 y + \nu_2)\ldots\ldots(a, b, c\ldots \between xy\, 1)^{m-2} = 0 \ldots\ldots(1).$$

Hence the points in which it meets the axis of x are given by

$$\lambda_1 \lambda_2 \ldots\ldots \lambda_m x^m + \ldots\ldots + \mathrm{B}x + \mathrm{C} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where C is the constant term in (1).

But, if P, Q, ... denote the asymptotes, and X the axis of x,

$$\sin \mathrm{XP} = \lambda_1 (\lambda^2+\mu^2)^{-\frac{1}{2}};$$

therefore, if $x_1, x_2 \ldots$ be the roots of (2), we have

$$(-)\quad \mathrm{C} = \lambda_1\lambda_2\ldots\lambda m\, x_1 x_2 \ldots x_m = \sin \mathrm{XP} \sin \mathrm{XQ} \ldots \left\{(\lambda_1^2+\mu_1^2)(\lambda_2^2+\mu_2^2)\ldots\right\}^{\frac{1}{2}};$$

or, if o be the origin, and l, m,... the points of section of X,

$$ol \,.\, om \ldots\ldots \sin \mathrm{XP} \sin \mathrm{XQ} \ldots\ldots = (-)^m \frac{\mathrm{C}}{(\lambda_1^2+\mu_1^2 \ldots\ldots)^{\frac{1}{2}}}.$$

But $\mathrm{C} = 0$ is the condition that o may be on S, and may be written $(\mathrm{S}o) = 0$, and the denominator is the condition that i or j, the circular points, may be on the curve, since the coordinates of i may be taken as $x : y : 1 = 1 : (-)^{\frac{1}{2}} : 0$.

Hence $$ol \,.\, om \ldots\ldots \sin \mathrm{XP} \sin \mathrm{XQ} \ldots\ldots = (-)^m \frac{(\mathrm{S}o)}{[(\mathrm{S}i)\,(\mathrm{S}j)]^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots (3).$$

Now, the condition $(\mathrm{S}o) = 0$ may be found from the tangential equation $\Sigma = 0$ by finding the condition that two tangents through o should coincide. If the tangential equation be

$$\Sigma \equiv \mathrm{A}\lambda^n + \mathrm{B}\lambda^{n-1}\mu + \ldots\ldots + \mathrm{C}\mu^n + \mathrm{D}\nu^n + \ldots\ldots = 0,$$

the tangents through the origin will be found by putting $\nu = 0$, or by the equation $$\mathrm{A}\lambda^n + \mathrm{B}\lambda^{n-1}\mu + \ldots\ldots + \mathrm{C}\mu^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now, if L, M, N,...... be the n tangents, we have

$$\sin^2 \mathrm{LM} \sin^2 \mathrm{LN} \ldots\ldots = \frac{(\lambda_1\mu_2-\lambda_2\mu_1)^2\,(\lambda_1\mu_3-\lambda_3\mu_1)^2 \ldots\ldots}{[(\lambda_1^2+\mu_1^2)\,(\lambda_2^2+\mu_2^2)\ldots\ldots]^{n-1}},$$

the numerator is $(\mathrm{S}o)$, and the denominator equated to zero is the condition that the origin may be a focus, or that a tangent may pass through i or j. Now the tangential equation may be written

$$\Sigma \equiv (x_1\lambda + y_1\mu + \nu)\,(x_2\lambda + y_2\mu + \nu) \ldots - \kappa\,(\lambda^2+\mu^2)\,(\mathrm{A}', \mathrm{B}', \ldots\ldots \between \lambda\mu\nu)^{n-2} = 0,$$

and, for a line through i or j, we have $\lambda : \mu : \nu = 1 : (-1)^{\frac{1}{2}} : 0$, and therefore $\Sigma = 0$ becomes $[x_1 + y_1(-1)^{\frac{1}{2}}]\,[x_2 + y_2(-1)^{\frac{1}{2}}] \ldots\ldots = 0$;

or, when freed from imaginaries,

$$(x_1^2+y_1^2)\,(x_2^2+y_2^2) \ldots\ldots = 0; \text{ or } op^2 \,.\, oq^2 \,.\, or^2 \ldots\ldots = 0;$$

therefore $$\sin^2 \mathrm{LM} \sin^2 \mathrm{LN} \ldots\ldots = \frac{(\mathrm{S}o)}{[op \,.\, oq \,.\, or \ldots]^{2(n-1)}};$$

therefore $$(\mathrm{S}o) = [op \,.\, oq \,.\, or \ldots\ldots]^{2(n-1)} \sin^2 \mathrm{LM} \sin^2 \mathrm{LN} \ldots\ldots$$

Again $(\mathrm{S}i) = 0$ is the condition that two of the tangents through i coincide, and it is easily seen that $[(\mathrm{S}i)\,(\mathrm{S}j)]^{\frac{1}{2}}$ reduces to $pq^2 \,.\, pr^2 \,.\, qr^2 \ldots\ldots$

Hence, from (3), we have

$$ol \,.\, om \ldots \sin \mathrm{XP} \sin \mathrm{XQ} \ldots = (-)^m \frac{[op \,.\, oq \,.\, or \ldots]^{2(n-1)} \sin^2 \mathrm{LM} \sin^2 \mathrm{LN}}{pq^2 \,.\, qr^2 \ldots\ldots}.$$

2. Again, from (2), $\lambda_1\lambda_2 \ldots\ldots \lambda_m x^m + \ldots\ldots + \mathrm{B}x + \mathrm{C} = 0$,

we have, if $x_1\, x_2, \ldots\ldots$ be the roots,

$$lm^2 \,.\, ln^2 \,.\, mn^2 \ldots\ldots = (x_1-x_2)^2\,(x_1-x_3)^2\,(x_2-x_3)^2 \ldots\ldots$$

But, if we form the condition $(\Sigma \mathrm{X}) = 0$, that (2) may have equal roots, we have $$(\Sigma \mathrm{X}) \equiv (\lambda_1\lambda_2 \ldots\ldots \lambda_m)^{2(m-1)}\,(x_1-x_2)^2\,(x_1-x_3)^2 \ldots\ldots;$$

therefore $$lm^2 . ln^2 . mn^2 = \frac{(\Sigma X)}{(\lambda_1 \lambda_2 \lambda_3 ......)^{2(m-1)}}.$$

Again, $$\sin XP \sin XQ ...... = \frac{\lambda_1 \lambda_2 ...... \lambda_m}{[(\lambda_1^2+\mu_2^2)(...)......]^{\frac{1}{2}}};$$ where λ_1, μ_1, &c. have the same values as in (2).

Therefore $$[\sin XP \sin XQ ...]^{2(m-1)} \, lm^2 . ln^2 . mn^2 ... = \frac{(\Sigma X)}{[(\lambda_1^2+\mu_1^2)...]^{m-1}},$$

and $$\sin^2 PQ \sin^2 QR ...... = \frac{(\lambda_1\mu_2-\lambda_2\mu_1)^2......}{[(\lambda_1^2+\mu_1^2)(......)]^{m-1}};$$

$$\therefore \frac{[\sin XP \sin XQ ...]^{2(m-1)} \, lm^2 . ln^2 . mn^2}{\sin^2 PQ \sin^2 QR} = \frac{(\Sigma X)}{(\lambda_1\mu_2-\lambda_2\mu_1)^2...} = \frac{(\Sigma X)}{(\Sigma I)},$$

where I is the line at infinity; since the denominator is the condition that I may be tangent to the curve.

Again, if we write the tangential equation in the form

$$C(x_1\lambda+y_1\mu+\nu)(x_2\lambda+y_2\mu+\nu)...+(\lambda^2+\mu^2)(A'\lambda^{n-2}+...+B\mu^{n-2}+...=0,$$

we have to determine the tangents through the origin,

$$[C x_1 x_2 ..... x^n + A']\lambda^n + \;......\; + [C y_1 y_2 ...... y_n + B']\mu^n = 0;$$

$$\therefore \sin XL \sin XM ... = \frac{\lambda_1 \lambda_2 ......}{[(\lambda_1^2+\mu_1^2)...]^{\frac{1}{2}}} = \frac{C y_1 y_2 ...... y_n + B'}{[(\Sigma oi)(\Sigma oj)]^{\frac{1}{2}}} = \frac{(\Sigma X)}{[(\Sigma oi)(\Sigma oj)]^{\frac{1}{2}}},$$

But $$\Sigma oi = C[x_1+y_1(-1)]^{\frac{1}{2}}[(x_2+y^2(-1)^{\frac{1}{2}}).. ...$$

$$\Sigma oj = C[x_1-y(-1)^{\frac{1}{2}}]......$$

therefore $$[(\Sigma oi)(\Sigma oj)]^{\frac{1}{2}} = C(op^2 . oq^2)^{\frac{1}{2}}...... = C\, op . oq ......;$$

therefore $$\sin XL \sin XM ...... op . oq ...... = \frac{\Sigma X}{C} = \frac{\Sigma X}{\Sigma I}$$

$$= \frac{(\sin XP \sin XQ ......)^{(2m-1)} \, lm^2 . ln^2 . mn^2 ......}{\sin^2 PQ \sin^2 QR ......}.$$

[A solution in the particular case of the conic is given on p. 99 of Vol. XII. of the *Reprint*.]

4707. (By W. S. Burnside, M.A.)—Eliminate x, y, z between

$$a_1x^2+b_1y^2+c_1z^2+2(f_1-\lambda)yz+2g_1zx+2h_1xy = 0,$$
$$a_2x^2+b_2y^2+c_2z^2+2f_2yz+2(g_2-\lambda)zx+2h_2xy = 0,$$
$$a_3x^2+b_3y^2+c_3z^2+2f_3yz+2g_3zx+2(h_3-\lambda)xy = 0,$$

and determine the degree of λ in the eliminant.

Solution by J. J. Walker, M.A.

It is well known that the variables may be eliminated linearly among the three given equations and the differential coefficients of their Jacobian

with respect to x, y, z. On writing down this eliminant in the form of a determinant, it is readily seen that the highest power of λ in it is λ^9, and that the coefficient of λ^9 is

$$8\left\{\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix} + \begin{vmatrix} 3a_1, & -b_1, & -c_1 \\ -a_2, & 3b_2, & -c_2 \\ -a_3, & -b_3, & 3c^3 \end{vmatrix}\right\} \text{ or } 32\,(7a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1).$$

ON A GENERALIZATION OF POLAR RECIPROCALS. *By* T. COTTERILL, M.A.

Let $x^2+y^2=k^2$ be the equation to a fixed circle, centre Z. Then $fx+gy=k^2$ is the polar of the point (f, g), and $\rho Z=k^2$, if ρ, Z be the respective distances of the point and its polar from the centre Z. Hence the polar reciprocal of a curve $F(\rho, \theta)=0$ in point polar coordinates becomes $F\left(\frac{k^2}{Z}, \theta\right)=0$ in line polar coordinates.

Now
$$(x-f)^2+(y-g)^2-(a-f)^2-(b-g)^2$$
$$=x^2+y^2-2f(x-a)-2g(y-b)-a^2-b^2=0$$
is the circle, centre (f, g), passing through the point S (a, b) and
$$f(x-a)+g(y-b)=\tfrac{1}{2}(k^2-a^2-b^2)=\delta^2, \text{ suppose,}$$
is the radical axis of the two circles as well as the polar of the point $(a+f, b+g)$ to the circle $\quad (x-a)^2+(y-b)^2=\delta^2$.

Also, ρ remaining as before, and S being the perpendicular from the point S on the line, we shall have $\rho S=\delta^2$; so that the envelope of the radical axes of the fixed circle and the circles through S which have their centres on the curve $F(\rho, \theta)=0$ is given by the tangential equation $F\left(\frac{\delta^2}{S}, \theta\right)=0$, the polar reciprocal to the circle, centre S and radius δ, of the curve $F(\rho, \theta)=0$ translated through the distance ZS.

4726. (By Prof. TOWNSEND, F.R.S.)—A plane being supposed to revolve round either inflexional tangent at any point of a cubic surface, show that the harmonic polar of the point, with respect to the curve of section, generates a ruled quadric surface passing through the two inflexional tangents at the point, and degenerating into a cone when the two tangents coincide.

I. *Solution by the* PROPOSER.

The quadric in question is evidently the polar quadric of the point with respect to the cubic, and therefore &c.; that quadric, as is well known, passing, for a point on any surface, through the two inflexional tangents at the point, and degenerating into a cone when those tangents coincide. (See SALMON's *Geometry of Three Dimensions*, 3rd Ed., Art. 284.)

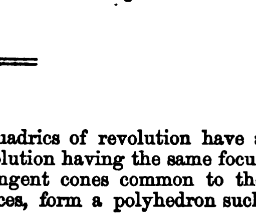
II. *Solution by the* Rev. F. D. THOMSON, M.A.

The two inflexional tangents at the point O, on the cubic surface, are the two tangents at the point which meet the surface in three consecutive points. Hence any section of the surface by a plane through an inflexional tangent will be a cubic curve of which the point O is a point of inflexion whose harmonic polar is a straight line. Now the locus of the extremities of harmonic means through O is a quadric touching the cubic at O, and the two generators through O are the two inflexional tangents to the cubic. Any plane through one of these generators will meet the quadric again in a straight line, which is, in fact, the harmonic polar of O with respect to the cubic curve of section. Hence the locus of the harmonic polar is the above quadric.

If the two inflexional tangents coincide, the two generators of the quadric coincide; in other words, it degenerates into a cone.

4716. (By R. W. GENESE, M.A.)—Through a fixed point any straight line is drawn meeting two given parallel lines in P and Q; through P, Q straight lines are drawn in fixed directions meeting in R; prove that the locus of R is a straight line.

Solution by C. B. S. CAVALLIN; R. F. DAVIS, B.A.; *and others.*



Let O be the fixed point, and let OM, ON, parallel to the fixed directions, meet the given parallel lines in M, N; then MN will be the required locus. For draw PR parallel to OM, meeting MN in R; and join QR; also let OM, ON meet NP, MQ in S, T. Then we have

$$MR : RN = SP : PN = MQ : QT;$$

hence QR is parallel to ON; therefore, &c.

4739. (By R. F. SCOTT, M.A.)—If n quadrics of revolution have a common focus, and a variable quadric of revolution having the same focus be drawn; prove that the vertices of the tangent cones common to the variable surface and each of the fixed surfaces, form a polyhedron such that all its edges pass through fixed points.

Solution by E. B. ELLIOTT, B.A.

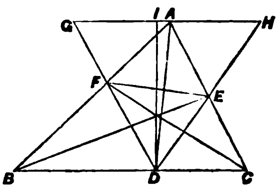
Reciprocating with regard to the common focus, we have to prove that,—If there be n fixed spheres, and another variable sphere be taken, the

planes of intersection of this last with the fixed spheres shall form a polyhedron whose edges lie in fixed planes.

This is easily seen. For let S_1, S_2, &c., S_n be the fixed spheres, and S the variable one. Then clearly the planes of intersection of S with the others, viz., $S-S_1$, $S-S_2$, &c., $S-S_n$ intersect two and two in lines each of which lies in one of the fixed planes S_1-S_2, S_1-S_3, S_2-S_3, &c.

4686. (By J. J. Walker, M.A.)—If the bisectors of the angles A, B, C of a plane triangle meet the opposite sides in D, E, F, and if θ be an angle such that $\tan\frac{1}{2}\theta = \frac{1}{2}\sec\frac{1}{2}A$; prove (1) that $\tan EDF = \tan\theta\cos\frac{1}{2}(B-C)$; and (2) that, if DE, DF meet a parallel to the base through A in G, H respectively, the segment GH is bisected at A, and is equal to the harmonic mean between CA and AB.

I. *Solution by* J. S. Jenkins.



We have $\dfrac{AH}{CD} = \dfrac{AE}{CE} = \dfrac{AB}{BC}$;

whence $AH \cdot BC = AB \cdot CD$;

similarly, $AG \cdot BC = AC \cdot BD$;

therefore $\dfrac{AH}{AG} = \dfrac{AB \cdot CD}{AC \cdot BD} = \dfrac{BD}{CD}\,\dfrac{CD}{BD} = 1$;

therefore $AH = AG$,

which proves that segment GH is bisected at A.

Again, $\dfrac{AG}{AF} = \dfrac{BD}{BF}$; therefore $AG = \dfrac{AF}{BF} \cdot BD$.

Similarly, $AH = \dfrac{AE}{CE} \cdot CD$.

And
$$GH = AG + AH = \frac{AF}{BF} \cdot BD + \frac{AE}{CE} \cdot CD$$
$$= BD \cdot \frac{AC}{BC} + CD \cdot \frac{AB}{BC} = AC \cdot \frac{BD}{BC} + AB \cdot \frac{CD}{BC}$$
$$= AC \cdot \frac{BC}{AB+AC} + \frac{AB \cdot AC}{AB+AC} = \frac{2AB.AC}{AB+AC},$$

which proves that GH is an harmonic mean between AB and AC.

Now, drawing DI perpendicular to GH, we have
$$\tan EDF = \tan HDG = \frac{GH \cdot DI}{DI^2 - EI \cdot HI} = \frac{GH \cdot DI}{AD^2 - AH^2} \quad \ldots\ldots(\alpha),$$
and
$$\tan\theta = \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta} = \frac{4\cos\frac{1}{2}A}{4\cos^2\frac{1}{2}A - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$
and $AD\sin\frac{1}{2}A\,(AB+AC) = \sin A\,(AB \cdot AC) = 2\sin\frac{1}{2}A\cos\frac{1}{2}A\,(AB \cdot AC)$;

whence $$\cos\tfrac{1}{2}A = AD\,.\,\frac{AB+AC}{2\,AB\,.\,AC} = \frac{AD}{GH} = \frac{AD}{2AH};$$

therefore $4\cos^2\tfrac{1}{2}A = \dfrac{AD^2}{AH^2}$, and therefore $\dfrac{AH^2}{AD^2-AH^2} = \dfrac{1}{4\cos^2\frac{1}{2}A-1}$;

whence $$4\cos\tfrac{1}{2}A\,.\,\frac{AH^2}{AD^2-AH^2} = \frac{4\cos\frac{1}{2}A}{4\cos^2\frac{1}{2}A-1}.$$

But, from (β), this expression is equal to $\tan\theta$,

therefore $$\tan\theta\cos\tfrac{1}{2}(B-C) = 4\cos\tfrac{1}{2}A\,\frac{AH^2}{AD^2-AH^2}\cos\tfrac{1}{2}(B-C)$$

$$= 2\,(\sin B+\sin C)\frac{AH^2}{AD^2-AH^2} = 2\,DI\left(\frac{AB+AC}{AB\,.\,AC}\right)\frac{AH^2}{AD^2-AH^2}$$

$$= \frac{4\,DI}{GH}\left(\frac{AH^2}{AD^2-AH^2}\right) = \frac{DI\,.\,GH^2}{GH\,(AD^2-AH^2)} = \frac{DI\,.\,GH}{AD^2-AH^2}.$$

And, from (α) this last expression is equal to $\tan EDF$;

therefore $$\tan EDF = \tan\theta\cos\tfrac{1}{2}(B-C).$$

II. *Solution by the* PROPOSER.

$$\frac{AG}{BD} = \frac{AF}{FB} = \frac{CA}{BC},\ \text{therefore}\ AG = \frac{CA\,.\,BD}{BC} = \frac{CA\,.\,AB}{CA+AB} = AH;$$

hence we have $$\frac{2}{GH} = \frac{1}{AB}+\frac{1}{CA}.$$

Therefore, drawing DK perpendicular to AD, and meeting AB in K,

$$AK = GH,\ \text{therefore}\ \tfrac{1}{2}\sec\frac{A}{2} = \frac{AG}{AD},\ \text{therefore}\ \tan\theta = \frac{GH\,.\,AD}{AD^2-AG^2}.$$

$$\text{Again, }\tan EDF = \frac{GH\sin G}{DG-GH\cos G} = \frac{2DG\,.\,GH\sin G}{2DG^2-2DG\,.\,GH\cos G}$$

$$= \frac{2GH\,.\,AC\sin C}{DG^2+DH^2-GH^2} = \frac{GH\,.\,AD\sin ADC}{AD^2-AG^2} = \tan\theta\,.\cos\tfrac{1}{2}(B-C).$$

4453. (By W. HOGG, M.A.)—A particle being placed at a given distance from a thin circular lamina of uniform density, in a line passing through its centre and perpendicular to its plane; find the velocity which it will acquire by moving to the lamina, the attractive force of each molecule of the lamina varying inversely as the square of the distance.

Solution by A. B. EVANS, M.A.; G. S. CARR; W. SIVERLEY; *and others.*

Let a be the radius of the circular lamina, k its thickness, ρ its density, b the given distance, and v the velocity required. Draw from the

centre of the lamina two adjacent concentric circles, the one with radius r, the other with radius $r+dr$. The mass of the circular ring included between these circles is $2\pi\rho\kappa r dr$, and the distance of every particle of this ring from the attracted particle is $(b^2+r^2)^{\frac{1}{2}}$.

The unit of attraction being the unit of mass at a unit's distance, the resultant attraction of the ring is $\dfrac{2\pi\rho\kappa r dr}{b^2+r^2}\cdot\dfrac{b}{(b^2+r^2)^{\frac{1}{2}}}$, the factor $\dfrac{b}{(b^2+r^2)^{\frac{1}{2}}}$, being the multiplier, in order to resolve the attraction of any element of the ring along the normal to the lamina through its centre. The resultant attraction of the whole lamina through its centre. The resultant attraction of the whole lamina is

$$2\pi\rho\kappa b\int_0^a \frac{r dr}{(b^2+r^2)^{\frac{3}{2}}} = 2\pi\rho\kappa\left\{1-\frac{b}{(b^2+a^2)^{\frac{1}{2}}}\right\}.$$

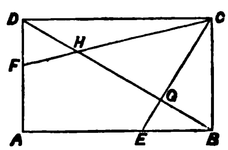
The equation $vdv=-fdx$ becomes, since

$$f=\pi\rho\kappa\left\{1-\frac{x}{(x^2+a^2)^{\frac{1}{2}}}\right\},\quad vdv=2\pi\rho\kappa\left\{\frac{x}{(x^2+a^2)^{\frac{1}{2}}}\right\}dx.$$

Observing that, when $x=b$, $v=0$, we have $v^2=4\pi\rho\kappa\left\{a+b-(a^2+b^2)^{\frac{1}{2}}\right\}$.

4714. (By C. Leudesdorf, B.A.)—ABCD is a given rectangle. On AB, AD respectively are taken two points E, F at random; CE, CF meet BD in G and H. Find the chance that the area of the triangle CGH shall be greater than one-fourth of the rectangle.

Solution by the Proposer.



Taking AB, AD as axes, let $AB=a$, $AD=b$, $AE=x$, $AF=y$; then G and H are

$$\left(\frac{a^2}{2a-x},\ \frac{b(a-x)}{2a-x}\right),\quad \left(\frac{a(b-y)}{2b-y},\ \frac{b^2}{2b-y}\right).$$

Therefore $\triangle\, CGH = \dfrac{ab}{2}\,\dfrac{ay+bx-xy}{(2a-x)(2b-y)}$.

If this be $>\frac{1}{4}ab$, we must have $y>4b\,\dfrac{a-x}{4a-3x}$; and as this value is always >0, and $<b$, the probability required is

$$\int_0^a\int_{4b\frac{a-x}{4a-3x}}^{b} dx\,dy \div \int_0^a\int_0^b dx\,dy$$

$$=\frac{1}{a}\int_0^a \frac{x\,dx}{4a-3x} = \tfrac{4}{9}\log_e 2-\tfrac{1}{3} = \cdot 283 \text{ nearly}.$$

4781. (By C. B. S. Cavallin.)—Show that, if three points are taken at random on the circumference of an ellipse, it is certain that a triangle can be constructed with the radii of curvature at these points as sides if the eccentricity of the ellipse be less than $(1-2^{-\frac{2}{3}})^{\frac{1}{2}}$.

Solution by C. Leudesdorf, B.A.

If the radii of curvature be ρ_1, ρ_2, ρ_3, we must have $\rho_1+\rho_2 > \rho_3$, and two similar inequalities. Now the maximum and minimum values of ρ_1, ρ_2, ρ_3 are $\frac{a^2}{b}$ and $\frac{b^2}{a}$ respectively; thus the problem is certainly possible if

$$\frac{b^2}{a}+\frac{b^2}{a} > \frac{a^2}{b};\ \text{ that is, if } \frac{b^3}{a^3} > \frac{1}{2};$$

that is, if $\qquad (1-e^2)^{\frac{3}{2}} > 2^{-1}$; or if $e < (1-2^{-\frac{2}{3}})^{\frac{1}{2}}$.

4695. (By Prof. Townsend, F.R.S.)—In a tricuspidal quartic curve, shew that the three conics passing through the intersection of the three cuspidal tangents, and touching each a pair of sides of the cuspidal triangle at the pair of opposite cusps, intersect the curve at the six vertices of two inscribed triangles whose sides pass in pairs through the corresponding cuspidal points.

I. *Solution by* R. W. Genese, M.A.

If $u=0$, $v=0$, $w=0$ be the equations to the sides of the cuspidal triangle multiplied by suitable constants, the equation to the quartic is

$$\pm u^{-\frac{1}{2}} \pm v^{-\frac{1}{2}} \pm w^{-\frac{1}{2}} = 0 \quad\text{.......................... (1).}$$

The intersection of the cuspidal tangents will be $u=v=w$, and the conics in question $\qquad u^2=vw,\ v^2=wu,$ and $w^2=uv$ .... (2).

From (1), $u^{-1}=v^{-1}+w^{-1}\pm 2(vw)^{-\frac{1}{2}}$; and, for intersection with $u^2=vw$, we have $u=w+v\pm 2(vw)^{\frac{1}{2}}$, therefore $w+v=-u$ or $3u$. The first gives imaginary values for the ratio $v:w$ (the six imaginary intersections of the conics and the quartic thus lie on $u+v+w=0$). The second gives

$$v:w=p:1 \text{ or } 1:p, \text{ where } p=\tfrac{1}{2}(7+3\sqrt{5});$$

therefore the points are $\frac{v}{p}=\frac{w}{1}=\frac{3u}{1+p},\ \frac{v}{1}=\frac{w}{p}=\frac{3u}{1+p}$;

or, as we may re-write them, if $\alpha=\sqrt{p}=\frac{1}{2}(3+\sqrt{5})$,

$$\frac{u}{\alpha}=\frac{v}{\alpha^2}=\frac{w}{1},\quad \frac{u}{\alpha}=\frac{v}{1}=\frac{w}{\alpha^2}.$$

The other points may be represented by

$$(1,\alpha,\alpha^2),\quad (\alpha^2,\alpha,1),\quad (\alpha^2,1,\alpha),\quad (1,\alpha^2,\alpha).$$

Let these be called in order AA′, BB′, CC′; then

$$\left.\begin{array}{llll} \text{A, B} & \text{lie on} & v = ua \\ \text{B, C} & \text{,,} & w = va \\ \text{C, A} & \text{,,} & u = va \end{array}\right\} \begin{array}{llll} \text{A}', \text{B}' & \text{lie on} & u = av \\ \text{B}', \text{C}' & \text{,,} & v = aw \\ \text{C}', \text{A}' & \text{,,} & w = au. \end{array}$$

Thus the sides of ABC, A′B′C′ pass in pairs through the cuspidal points.

II. *Solution by the* PROPOSER.

Denoting by x, y, z the ratios of the trilinear coordinates of an arbitrary point P to those of the cuspidal centre O with respect to the three sides BC, CA, AB of the cuspidal triangle ABC, then the equations of the three branches of the curve standing on the three chords BC, CA, AB, respectively, being

$$y^{-\frac{1}{2}}+z^{-\frac{1}{2}}=x^{-\frac{1}{2}},\ \ z^{-\frac{1}{2}}+x^{-\frac{1}{2}}=y^{-\frac{1}{2}},\ \ x^{-\frac{1}{2}}+y^{-\frac{1}{2}}=z^{-\frac{1}{2}} \ \ldots\ldots\ldots\ldots \ (1),$$

and those of the three corresponding conics, passing through O, and standing on the same chords, being

$$yz = x^2,\ \ zx = y^2,\ \ xy = z^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2),$$

if A_1 and A_2, B_1 and B_2, C_1 and C_2 be the three pairs of points at which the latter intersect the non-corresponding pairs of the former, respectively, we get at once, for their coordinate ratios, the two triads of relations

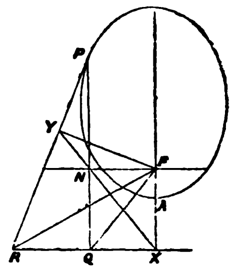
$$\left.\begin{array}{l} \left(\frac{y}{x}\right)_{A_1}+\left(\frac{y}{x}\right)_{A_1}^{\frac{1}{2}}=1,\ \ \left(\frac{z}{y}\right)_{B_1}+\left(\frac{z}{y}\right)_{B_1}^{\frac{1}{2}}=1,\ \ \left(\frac{x}{z}\right)_{C_1}+\left(\frac{x}{z}\right)_{C_1}^{\frac{1}{2}}=1 \\ \left(\frac{z}{x}\right)_{A_2}+\left(\frac{z}{x}\right)_{A_2}^{\frac{1}{2}}=1,\ \ \left(\frac{x}{y}\right)_{B_2}+\left(\frac{x}{y}\right)_{B_2}^{\frac{1}{2}}=1,\ \ \left(\frac{y}{z}\right)_{C_2}+\left(\frac{y}{z}\right)_{C_2}^{\frac{1}{2}}=1 \end{array}\right\} \ldots \ (3),$$

from the first of which the collinearity of B_1 and C_1 with A, of C_1 and A_1 with B, and of A_1 and B_1 with C, and from the second of which that of B_2 and C_2 with A, of C_2 and A_2 with B, and of A_2 and B_2 with C, following immediately by virtue of equations (2), therefore, &c.

COR.—The following construction by elementary geometry for the solution of the problem—"Given, of a tricuspidal quartic curve, the three cuspidal points A, B, C, and the cuspidal centre O, to determine the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ whose vertices shall lie on the curve, and whose sides shall pass through the cuspidal points,"—is evident from equations (3) above; viz., drawing through the vertex of each angle of the cuspidal triangle the two lines which with the corresponding cuspidal tangent divide the angle in the two reciprocal anharmonic ratios given by the roots of any one of the six equations (3), the six lines so drawn are the sides of the two triangles required.

4731. (By R. W. GENESE.)—**In any conic, if N be the foot of the perpendicular from any point P of the curve on the latus rectum, Y the foot of the perpendicular from a focus on the tangent at P; then NY will pass through a fixed point (viz., the foot of the corresponding directrix).**

I. *Solution by* S. A. RENSHAW, NILKANTA SARKAR, *and others.*



Let the tangent meet the directrix in R and PN meet it in Q; join FR, FQ; then a circle may be drawn through F, P, R, Q, and also through F, P, Y, N;

therefore $\angle$ PRF = PQF = NXF,

and PRF = YFP = YNP;

therefore NXF = YNP,

and PN, FX are parallel; therefore YNX is a straight line, or, in other words, YN passes through X.

II. *Solution by* C. LEUDESDORF, B.A.

As in Question 4647 (p. 46 of this volume) the equation to NY is found to be

$$u \cos\alpha = f(\alpha)\cos\theta + f'(\alpha)\sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Now $\quad u = f(\theta) = \dfrac{1+e\sin\theta}{c}$, therefore $f'(\alpha) = +\dfrac{e}{c}\cos\alpha$,

and (1) becomes $\quad \cos\alpha\left(u - \dfrac{e}{c}\sin\theta\right) = \dfrac{1+e\sin\alpha}{c}\cos\theta$,

which always passes through the fixed point $\theta = \dfrac{\pi}{2}$, $u = \dfrac{e}{c}$.

4183. (By S. FORDE, M.A.)—A heavy particle is projected upwards from a point A with a given velocity, and in a given vertical plane, but in any direction taken at random. At some point P of its orbit, taken at random between A and B, its velocity is suddenly diminished by one mth of its value. Find the average *vis viva* of the particle t seconds after leaving P. And if m be very large, find its value in order that when $t = \frac{2}{3}\sqrt{3}$, the average found may be two-thirds of the *vis viva* of the particle at the moment of starting from A.

Solution by the PROPOSER.

Let the direction of projection make $\angle\theta$ with AB, and let V be the velocity of projection. Let P be given by

$$x = Vt\cos\theta, \quad y = Vt\sin\theta - \tfrac{1}{2}gt^2.$$

Then, if V′ be the new velocity at P, we have

$$V'^2 = (1-m^{-1})^2(V^2 + g^2t^2 - 2gVt\sin\theta);$$

therefore, if v' = velocity T seconds after leaving P, θ the angle made by the tangent at P with AB, we have

$$\begin{aligned} v'^2 &= V'^2 + g^2T^2 - 2gV'T\sin\theta' \\ &= (1-m^{-1})^2(V^2 + g^2t^2 - 2gVt\sin\theta) - 2gT(1-m^{-1})(V\sin\theta - gt) + g^2T^2; \end{aligned}$$

therefore, taking the mass of the particle as unity, the average *vis viva*

$$= \frac{1}{\pi}\int_0^\pi \left[\int_0^{\frac{2V\sin\theta}{g}} \frac{g}{2V\sin\theta}\, v'^2\, dt\right] d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \left[\left(1-\frac{1}{m}\right)^2\left(V^2 \frac{2V^2\sin^2\theta}{3}\right) + g^2T^2\right] d\theta = \tfrac{2}{3}\left(1-\frac{1}{m}\right)^2 V^2 + g^2T^2.$$

If this be equal to $\frac{2}{3}V^2$, then $\frac{2}{3}V^2\left(\frac{2}{m}-\frac{1}{m^2}\right) = g^2T^2$; and if m be large, $\frac{1}{m^2}$ may be neglected, and $m = \frac{4V^2}{3g^2T^2} = \left(\frac{V}{g}\right)^2$ if $T = \frac{2}{\sqrt{3}}$.

4735. (By J. J. Walker, M.A.)—If lines drawn through a double point on a plane cubic curve make angles $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ with the tangents at the double point, the three points in which they again meet the cubic will be collinear, if the determinant

$$\begin{vmatrix} \sin^2\alpha, & \sin^2\beta, & \sin^2\gamma \\ \sin^2\alpha', & \sin^2\beta', & \sin^2\gamma' \\ \sin\alpha\sin\alpha', & \sin\beta\sin\beta', & \sin\gamma\sin\gamma' \end{vmatrix} \text{ vanishes.}$$

I. *Solution by the* Rev. F. D. Thomson, M.A.

The condition stated in the question is nugatory. In fact, the determinant reduces to

$$\left(\frac{\sin\alpha}{\sin\alpha'} - \frac{\sin\beta}{\sin\beta'}\right)\left(\frac{\sin\beta}{\sin\beta'} - \frac{\sin\gamma}{\sin\gamma'}\right)\left(\frac{\sin\gamma}{\sin\gamma'} - \frac{\sin\alpha}{\sin\alpha'}\right) = 0,$$

which can only be satisfied by two of the three lines coinciding. The condition for the three points being collinear depends upon the particular cubic which may be drawn with the given double point and tangents.

For a given cubic the condition may be found as follows. The equation to a cubic having the origin for a double point and the axes for tangents, may be written $(a, b, c, d \between x, y)^3 + xy = 0$. Let $Ax + By = 1$ be any transversal, then for the points of intersection we have the homogeneous equation $\quad (a, b, c, d \between x, y)^3 + xy\,(Ax + By) = 0$,

or if $y = \lambda x$, $\quad (a, b, c, d \between 1, \lambda)^3 + \lambda\,(A + B\lambda) = 0$,

therefore, if λ, μ, ν be the three roots of this equation, $d\lambda\mu\nu + a = 0$, the condition required, which is independent of the particular transversal.

II. *Solution by the* Proposer.

Mr. Thomson's remark as to the condition stated in the Question being nugatory, is at once seen to be correct. It is, in fact, the irrelevant factor

in the investigation of the condition which might *à priori* be expected. But Mr. Thomson does not obtain "the condition for the three points being collinear" by his method; he merely proves that, if the three points are collinear, the relation $d\lambda\mu\nu + a = 0$ holds.

The converse proposition, due, I believe, to Herr Weyr of Prague, may however be proved thus: The condition that the lines $x = \lambda y$, $x = \mu y$, $x = \nu y$, should meet the curve $(abcd \between xy)^3 + xy = 0$ in three finite collinear points is $\begin{vmatrix} \lambda & \mu & \nu \\ \lambda^2 & \mu^2 & \nu^2 \\ \mathrm{L} & \mathrm{M} & \mathrm{N} \end{vmatrix} = 0$, where $\mathrm{L} = (abcd \between 1\lambda)^3$, $\mathrm{M} = (abcd \between 1\mu)^3$, $\mathrm{N} = (abcd \between 1\nu)^3$.

But the above determinant is identically equal to $(a + d\lambda\mu\nu) \begin{vmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ \lambda^2 & \mu^2 & \nu^2 \end{vmatrix}$

i.e., is independent of b, c. Hence, rejecting the nugatory or irrelevant condition, $a + d\lambda\mu\nu = 0$ is the necessary and sufficient condition.

4793. (By Professor Wolstenholme, M.A.)—If $y = x^n (\log x)^r$, n, r being integers, prove that

$$x_r \frac{d^{n+r}y}{dx^{n+r}} + \frac{r(r-1)}{|\underline{2}} x^{r-1} \frac{d^{n+2-1}y}{dx^{n+2-1}} + \frac{r(r-1)(r-2)(3r-5)}{24} x^{r-2} \frac{d^{n+r-2}y}{dx^{n+r-2}} + \dots$$
$$+ (2^{r-1}-1)x^2 \frac{d^{n+2}y}{dx^{n+2}} + x\frac{d^{n+1}y}{dx^{n+1}} = |\underline{r}\ |\underline{n},$$

the coefficients being

$$\frac{\Delta^{r-1}1^{r-1}}{|\underline{r-1}}, \quad \frac{\Delta^{r-2}1^{r-1}}{|\underline{r-2}}, \quad \frac{\Delta^{r-3}1^{r-1}}{|\underline{r-3}}, \quad \dots\dots \quad \frac{\Delta 1^{r-1}}{|\underline{1}}, \text{ and } 1;$$

so that the result may be symbolically written

$$\epsilon^{x\mathrm{D}\Delta}\left(1^{r-1}\frac{d^{n+1}y}{dx^{n+1}}\right) = \frac{|\underline{r}\cdot|\underline{n}}{x},$$

where D denotes $\dfrac{d}{dx}$ and operates on $\dfrac{d^n y}{dx^n}$ only, and Δ operates on 1^{r-1} only, the terms after the rth all vanishing since $\Delta^m x^n = 0$, when m is an integer $> n$. The calculations involved prove that, when $x = 1$,

$$\Delta^{n-1}x = \frac{|\underline{n+1}}{2}, \quad \Delta^{n-2}x^n = |\underline{n+1}\,.\,\frac{(3n-2)}{24},$$

$$\Delta^{n-3}x^n = |\underline{n+1}\,.\,\frac{(n-1)(n-2)}{48}.$$

Solution by Professor Cayley.

Since $y = x^n (\log x)^r$, therefore $(xd_x - n)\,y = rx^n (\log x)^{r-1}$; and, by repeating the same operation, we have

$$(xd_x - n)^r y = [r]^r x^n; \text{ whence } d_x^n (xd_x - n)^r y = [r]^r [n]^n.$$

Now, for any value whatever of the function y, we have

$$d_x^n (xd_x-n)^r y = \text{A}x^r d_x^{r+n} y + \text{B}x^{r-1} d_x^{r+n-1} y + \text{C}x^{r-2} d_x^{r+n-2} y + \&\text{c.},$$

the coefficients A, B, C ... being functions, presumably of r, n, but independent of the form of the function y. It will, however, appear that A, B, C ... are, in fact, functions of r only.

To see how this is, observe that $(xd_x-n)^r$ consists of a set of terms $(xd_x)^\theta$, ($\theta=0$ to r,) where $(xd_x)^\theta$ denotes θ repetitions of the operation xd_x; and by a well known theorem this is $=[xd_x+\theta-1]^\theta$, where, after expansion of the factorial, $(xd_x)^\theta$ is to be replaced by $x^\theta d_x^\theta$, thus $(xd_x)^2 = [xd_x+1]^2$, $= x^2d_x^2 + xd_x$, $(xd_x)^3 = [xd_x+2]^3$, $= x^3d_x^3 + 3x^2d_x^2 + 2xd_x$, &c.; thus $(xd_x-n)^r$ consists of a series of terms $x^\theta d_x^\theta$, ($\theta=0$ to r,) and, operating with d_x^n, this last, $= (d_x+d_x')^n$, consists of a series of terms such as $d_x^\alpha d_x'^{\,n-\alpha}$, where the unaccented symbol operates on the x^θ, and the accented symbol on the y; the term is thus $x^{\theta-\alpha} d_x^{n+\theta-\alpha}$, or observing that $\theta-\alpha$ is at most $=r$, and putting it $=r-k$, the term is $x^{r-k} d_x^{n+k}$, viz. $d_x^n (xd_x-n)^r$ consists of a series of terms of the form $x^{r-k} d_x^{n+k}$; or, what is the same thing, $d_x^n (xd_x-n)^r y$ is a series of the form in question.

To understand how it can be that the coefficients A, B, C ... are independent of n, take the particular case $r=2$; then we have here

$$d_x^n (xd_x-n)^2 y = \text{A}x^2 d_x^{n+2} y + \text{B}x d_x^{n+1} y + \text{C}d_x^n y;$$

and the right-hand side is $d_x^n \left\{x^2d_x^2 - (2n-1)\, xd_x + n^2\right\} y$,

$$\begin{aligned} \text{which is} \quad &= \left\{x^2 d^{n+2} + 2nxd_x^{n+1} + (n^2-n)\, d_x^n\right\} y \\ &- (2n-1)\left\{ \qquad xd_x^{n+1} + \qquad nd_x^n \right\} y \\ &+ n^2 \left\{ \qquad\qquad\qquad\qquad d_x^n \right\} y; \end{aligned}$$

hence $\text{A}=1$, $\text{B} = 2n-(2n-1)$, $=1$, $\text{C} = (n^2-n) - n\,(2n-1) + n^2$, $=0$; and we thus see also how in this particular case the last coefficient is $=0$, viz., that we have $d_x^n (xd_x-n)^2 y = x^2 d_x^{n+2} y + x d_x^{n+1} y$,

without any term in $d_x^n y$.

To find the coefficients A, B, C ... generally, write $y = x^{r+n+\theta}$, then $xd_x - n = r+\theta$, and consequently

$$d_x^n (xd_x-n)^r y, \; = (r+\theta)^r d_x^n x^{r+n+\theta}, \; = (r+\theta)^r [r+n+\theta]^n x^{r+\theta};$$

whence $(r+\theta)^r [r+n+\theta]^n = \text{A}\,[r+\theta+n]^{n+r} + \text{B}\,[r+\theta+n]^{n+r-1} + \ldots\ldots;$

or, what is the same thing, $(r+\theta)^r = \text{A}\,[r+\theta]^r + \text{B}\,[r+\theta]^{r-1} + \ldots\ldots,$

and since the left-hand side, and every term $[r+\theta]^s$ on the right-hand side, contains the factor $r+\theta$, there is not on the right-hand side any term $[r+\theta]^0$; dividing the equation by $r+\theta$, it then becomes

$$(r+\theta)^{r-1} = \text{A}\,[r+\theta-1]^{r-1} + \text{B}\,[r+\theta-1]^{r-2} + \ldots\ldots,$$

and we thus have $\text{A} = \dfrac{\Delta^{r-1}1^{r-1}}{[r-1]^{r-1}}\;(=1)$, $\text{B} = \dfrac{\Delta^{r-2}1^{r-1}}{[r-2]^{r-2}}$;

viz., writing $r+\theta = 1+x$, $u_x = (1+x)^{r-1}$, and taking the terms in the reverse order, the series is the well known one

$$u_x = u_0 + \frac{x}{1}\,\Delta u_0 + \frac{x\,.\,x-1}{1\,.\,2}\,\Delta^2 u_0 + \&c.$$

Hence, in general,

$$d_x^n\,(xd_x - n)^r\,y = \frac{\Delta^{r-1}\,1^{r-1}}{[r-1]^{r-1}}\,x^r\,d_x^{r+n}\,y + \frac{\Delta^{r-2}\,1^{r-1}}{[r-2]^{r-2}}\,x^{r-1}\,d_x^{r+n-1}\,y + \&c.,$$

where observe that the last term is $= xd_x^{n+1}\,y$; and for the function $y = x^n\,(\log x)^n$, the value of each side is $= [r]^r\,[n]^n$.

4791. (By Sir JAMES COCKLE, F.R.S.)—An elastic fluid ($p = A^2\rho$) is moving in a right circular cylinder of infinite length and, the motion being supposed to be all parallel to the axis of the cylinder and all the particles of the orthogonal disc whose abscissa is x to have the same velocity u and direction of motion, the system

$$Bt = \int_1^v \frac{dv}{(\log v)^{\frac{1}{2}}},\quad u = \frac{(\log v)^{\frac{1}{2}}}{v}\left\{B\,(x-at)+b\right\} + a \;\ldots\ldots\; (1,\,2),$$

$$\log\rho = -\frac{1}{4A^2}\left\{\frac{(u-a)^2}{\log v} - c^2\right\} - \log v \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3),$$

is a solution of the equations of the motion. Interpret these formulæ.

Solution by the PROPOSER.

1. Let $\left(\frac{c}{2A}\right)^2 = \log\lambda$ and $\frac{b}{B} = -\beta$, then (2), (3) may be written

$$u-a = \frac{(\log v)^{\frac{1}{2}}}{v}\,B\,(x-at-\beta) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (4),$$

$$\log\frac{v\rho}{\lambda} = \left(\frac{B}{2Av}\right)^2 (x-at-\beta)^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5),$$

and when $v=1$, and therefore $t=0$, we have $u=a$ throughout. Let $t=0$ give $x=X$ and $\rho=R$. Then we have

$$\log\frac{R}{\lambda} = -\left(\frac{B}{2A}\right)^2 (X-\beta)^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (6),$$

and $X=\beta$ gives $R=\lambda$.

2. Take x and β positively. Draw a straight line on the surface of the cylinder; and on this line, and at a distance β from a plane drawn through the origin perpendicular to the axis, place a material particle. When $t=0$, start this particle along the line with a velocity $= a$. Then the disc in whose plane the particle for the time being is situate, has the same

velocity (a) as the particle; for the abscissa of disc and particle at the time t is $= at+\beta$, and when $x = at+\beta$ then (4) gives $u = a$. The discs in advance of the particle will have a greater, and those behind it a less, velocity than the particle.

3. Let $a=0$, then u vanishes when $t=0$. Suppose that, when $t=0$, the fluid is held in equilibrium by a force F. Then, by (6), we have

$$F = \frac{A^2}{R}\frac{dR}{dX} = -\tfrac{1}{4}B\ (X-\beta) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (7),$$

or

$$F = -\tfrac{1}{4}B\,(BX+b) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (8).$$

Hence, if to the origin there tend a force $\frac{1}{4}B\,(BX+b)$, the fluid will be in equilibrium, and the distribution of the fluid, thus at rest, will be the same as that of the moving fluid when $t=0$. Consequently, if such fluid be in equilibrium under such force, and such force suddenly ceases to act, the motion of the liberated fluid will be determined by the system (1), (2) and (3), if in (2) and (3) we make $a=0$.

4. If we suppose v to be eliminated from (2) and (3) by means of (1), we shall have a complete solution of the equations of the motion. Again,

$$u-a = \frac{x-c}{t-b},\quad A^2\log\frac{\rho'}{\lambda} = (a^2-A^2)\log(t-b)-au \quad\ldots\ldots\ldots\ (8,\ 9)$$

will be another complete solution of such equations. Hence, if these solutions be incapable of being made coincident by any transformation, two simultaneous partial primordinals, containing two dependent and two independent variables, may admit of at least two independent complete solutions.

5. When generalization is not in question, we simplify the formulæ by putting $b=0$. When $v<1$ the system (1), (2) and (3) cannot be applied. At p. 223 of his *Leçons sur le Calcul des Fonctions* (1806), Lagrange says that zero divided by zero is, so to say, the means which analysis employs to escape from contradictions; imaginary roots do not indicate, properly speaking, a contradiction, but an impossibility (*ib.*)

4696. (By Prof. Clifford.)—Six circles pass through twelve points on a conic in the following order,

$$(a)\ldots A_1A_2A_3A_4,\quad (b)\ldots B_1B_2B_3B_4,\quad (c)\ldots C_1C_2C_3C_4,$$
$$(d)\ldots A_1A_2B_3C_4,\quad (e)\ldots B_1B_2C_3A_4,\quad (f)\ldots C_1C_2A_3B_4;$$

prove that two circles and another point may be taken arbitrarily, and that the circles abc meet the circles def in six new points which lie on the circumference of another circle.

Solution by the Proposer.

On pp. 42, 43 of this volume of the *Reprint*, Mr. F. D. Thomson solves the first part of this question, and adds, "the second part of the

proposition is not true." It may be interesting to compare the following demonstration with the grounds of this statement, if Mr. THOMSON will kindly give us them; whichever of the two should turn out to be valid.

Three circles taken together constitute a sextic curve passing three times through each of the circular points at infinity. Now the sextic *def* passes through all the twelve points of intersection of the sextic *abc* with the conic, which we may call k; hence, by a well-known theorem, there must be an identical equation of the form

$$\mu \,.\, def = \lambda \,.\, abc + q \,.\, k.$$

Here λ and μ are numerical ratios, and q is a quartic function of the coordinates. The equation may also be written

$$-q \,.\, k = \lambda \,.\, abc - \mu \,.\, def,$$

and in this form it shows that the equation $qk = 0$ represents a curve of the sixth order passing three times through each circular point. But, by hypothesis, the conic k does not pass through either circular point. The quartic curve q has therefore two triple points on the line infinity; it must therefore contain that line. The rest of it is a cubic having two double points on the line infinity; it also must therefore contain that line. The final remainder is a conic passing once through each circular point, that is to say, a circle. Calling this circle s, we reduce our equation to the form

$$s \,.\, k \,.\, \infty^2 = \lambda \,.\, abc - \mu \,.\, def,$$

which shows that the remaining six intersections of *abc* with *def* lie on a circle s.

It will be observed that the proof holds good if we substitute for the conic in the enunciation a circular cubic or a bicircular quartic. From the latter extension we may obtain a transformed theorem of some interest. Invert the whole figure in regard to a point not in its plane; the bicircular quartic becomes a section of a sphere by an arbitrary quadric surface, and every circle becomes a section of the sphere by a plane. In this form we may substitute for the sphere any quadric surface, and the transformed theorem may then be stated and proved as follows:—

If six planes pass through twelve points on a quadriquadric curve in the order above stated; the six lines of intersection ae, af, bf, bd, cd, ce will meet every quadric surface passing through the curve in six points which lie in one plane.

It is to be observed that these six lines already meet the quadriquadric curve in the six points A_3, A_4, B_3, B_4, C_3, C_4. Let h_2, k_2 be two quadric surfaces passing through the curve; then the cubic surface *abc* passes through all the twelve intersections of the cubic *def* and the quadrics h_2, k_2. We must therefore have an indentical equation of the form

$$\lambda \,.\, abc = \mu \,.\, def + uh_2 + vk_2,$$

where λ and μ are numerical ratios, while u and v are expressions of the first order in the coordinates. Writing this identity in the form

$$vk_2 = \lambda \,.\, abc - \mu \,.\, def - uh_2,$$

we see that the nine lines of intersection of *abc* and *def* must meet the quadric h either on the quadric k_2 (*i.e.*, on the quadriquadric curve) or on the plane v. Now three of these, *ad*, *be*, *cf* meet the curve in two points each, and the rest, *ae*, *af*, *bf*, *bd*, *cd*, *ce* in one point each; consequently these latter must meet the quadric h_2 on the plane v.

The construction of the figure depends first on that of the hexagon

$A_4A_3B_4B_3C_4C_3$. In the case of the plane conic the opposite sides of this hexagon are parallel, and the possibility of the construction is assured by Pascal's theorem. When the hexagon has been drawn, it is easy to make a pair of circles pass through the ends of two opposite sides and intersect on the conic. In this way I have drawn the figure as carefully as I can, and it seems to come right. In the case of the quadriquadric curve, each pair of opposite sides is such that a quadric surface can be drawn through them to contain the curve. In the first instance they are given as two chords which are both met by the same third chord; thus the lines A_4A_3, B_3C_4 are both met by A_1A_2. Now the problem, to draw a straight line meeting two given straight lines and a quadriquadric curve twice, admits in general of eight solutions; but in the case where the two given lines are chords of the curve, the four lines joining their points of intersection count for two solutions each, and if there is one other solution, there must be an infinite number; *i.e.*, the two lines and the curve must lie on the same quadric surface. The possibility of the inscription of a hexagon whose opposite sides possess this property may be shown by a method analogous to that which Mr. THOMSON has used for the plane conic. The quadriquadric is a curve of deficiency *one*, and therefore the coordinates of any point on it may be expressed as elliptic functions of a parameter; this may be so taken that the sum of the parameters of four points in one plane shall be congruent to zero (Clebsch, "On the application of Abel's functions to Geometry," *Crelle's Journal*). Using the letters A_1, A_2, etc., to represent these parameters, we shall have

$$A_1 + A_2 + A_3 + A_4 \equiv 0 \text{ (mod. } \omega, \omega', \text{ where } \omega, \omega' \text{ are the periods)},$$

$$A_1 + A_2 + B_3 + C_4 \equiv 0;$$

and therefore $A_3 + A_4 \equiv B_3 + C_4$ (mod. ω, ω'), as the condition to be satisfied by two opposite sides of the hexagon. Now, if this condition is satisfied by two pairs of opposite sides, it will be satisfied by the third pair; for the congruence $B_4 + B_3 \equiv C_3 + A_4$ follows from the congruences

$$A_4 + A_3 \equiv B_3 + C_4, \qquad C_4 + C_3 \equiv A_3 + B_4.$$

The theorem states that, when such a hexagon has been constructed, lines may be drawn through its vertices which shall meet every quadric surface passing through the curve in six points on one plane. *As the surface varies, this plane passes through a fixed line;* for

$$uh_2 + vk_2 = u\,(h_2 + \rho k_2) + (v - \rho u)\,k_2.$$

Lastly, I observe that not every skew hexagon can have a quadriquadric curve drawn through it so that each pair of opposite sides shall be generators of the same quadric passing through the curve. Let the hexagon $A_4A_3B_4B_3C_4C_3$ be given; through the lines A_4A_3, B_3C_4 and the points B_4, C_3, a singly infinite number of quadrics can be drawn, which will intersect in a quadriquadric curve; and one condition is necessary in order that the chords A_3B_4, C_4C_3 may possess the required property, or, which is the same thing, that the three curves which we may thus get from the three pairs of opposite sides may be identical. The hexagon therefore possesses a geometrical property which can doubtless be expressed in terms of its diagonal lines or planes; this expression, however, I have not as yet been able to find.

[MR. THOMSON is now of opinion, after seeing the above proof, that the second part of this beautiful theorem *is* true. He thinks he must have made some error of notation in drawing the somewhat complicated figure.]

3067. (By the Rev. J. Wolstenholme, M.A.)—

If $$a^3+b^3+c^3 = (b+c)(c+a)(a+b),$$
and $$(b^2+c^2-a^2)\,x = (c^2+a^2-b^2)\,y = (a^2+b^2-c^2)\,z,$$
prove that $$x^3+y^3+z^3 = (y+z)(z+x)(x+y).$$

Solution by J. J. Sylvester, F.R.S.

Call $F(a, b, c) = a^3+b^3+c^3-(a+b)(a+c)(b+c)$, and let
$$A = a^4-b^4-c^4+2b^2c^2,\quad B = b^4-a^4-c^4+2a^2c^2,\quad C = c^4-a^4-b^4+2a^2b^2.$$
If it can be shown that
$$F(a,b,c)\,F(-a,b,c)\,F(a,-b,c)\,F(a,b,-c)-F(A,B,C) = U = 0,$$
then Mr. Wolstenholme's theorem is evidently true.

Now U is a function of a^2 of the sixth degree, and is obviously 0 when $a^2=\infty$; if then $U=0$ for six other values of a^2, then U is always 0. Suppose then, successively, $a=0$, $a=b$, $a=c$, $a^2=b^2+c^2$, $a^2=-b^2+c^2$, $a^2=b^2-c^2$; these really amount (on account of the symmety of U) to the three distinct cases $a=o$, $a=b$, $a^2=b^2+c^2$; in each of these cases the calculations become extremely simple, and show that U is equal to 0. Hence $U=0$ is identically true, and the theorem of Mr. Wolstenholme is demonstrated. It is worthy of note that the derivative given by this theorem is not included in the general derivative of a, b, c, corresponding to the tangential of a, b, c, regarded as a point on the cubic curve $F(x, y, z)$, which tangential has for coordinates functions of the fourth degree in a, b, c, notwithstanding that the coordinates of Mr. Wolstenholme's derivative, expressed in an integral form, are also of the fourth degree.

4764. (By the Rev. F. D. Thomson, M.A.)—A heavy uniform rod, length $2a$, slips down with its extremities in contact with a smooth horizontal floor and a smooth vertical wall, not being initially in a plane perpendicular to the wall. Show that, if θ be the inclination to the vertical, ψ the inclination of the horizontal projection of the rod to the common section of the planes, the motion is determined by the equations
$$4\frac{d^2}{dt^2}(\cos\theta) = \cot\theta\sec\psi\frac{d^2}{dt^2}(\sin\theta\cos\psi)-3\frac{g}{a},$$
$$4\frac{d^2}{dt^2}(\sin\theta\sin\psi) = \tan\psi\frac{d^2}{dt^2}(\sin\theta\cos\psi),$$
and deduce a first integral.

Solution by S. Tebay, B.A.

Let the intersection of the planes be the axis of y, and let the plane xz pass through the initial position of the centre of gravity, which will be confined to this plane during the motion, since no lateral forces act upon the rod.

Let (x, y, z) be any point in the rod, and r its distance from the centre. Then $x = (a-r) \sin\theta \sin\psi,\ y = r \sin\theta \cos\psi,\ z = (a+r)\cos\theta.$

By D'Alembert's principle, and the principle of virtual velocities, we have $$\int dr \frac{d^2x}{dt^2}\,\delta x + \int dr \frac{d^2y}{dt^2}\,\delta y + \int dr \left(\frac{d^2z}{dt^2} + g\right)\delta z = 0.$$

Since the variations $d\theta$ and $d\psi$ are independent, we have

$$\cos\theta \sin\psi \int (a-r)\, dr \frac{d^2x}{dt^2} + \cos\theta\cos\psi \int r\, dr \frac{d^2y}{dt^2}$$
$$- \sin\theta \int (a+r)\, dr \left(\frac{d^2z}{dt^2} + g\right) = 0,$$
$$\sin\theta\cos\psi \int (a-r)\, dr \frac{d^2x}{dt^2} - \sin\theta\sin\psi \int r\, dr \frac{d^2y}{dt^2} = 0.$$

Substitute the values of $\frac{d^2x}{dt^2},\ \frac{d^2y}{dt^2},\ \frac{d^2z}{dt^2}$, and integrate with respect to r;

therefore $$4\cos\theta\sin\psi \frac{d^2}{dt^2}(\sin\theta\sin\psi) + \cos\theta\cos\psi \frac{d^2}{dt^2}(\sin\theta\cos\psi)$$
$$-4\sin\theta \frac{d^2}{dt^2}(\cos\theta) - 3\frac{g}{a}\sin\theta = 0,$$
$$4\sin\theta\cos\psi \frac{d^2}{dt^2}(\sin\theta\sin\psi) - \sin\theta\sin\psi \frac{d^2}{dt^2}(\sin\theta\cos\psi) = 0.$$

These equations give the proposed results. Multiply the first by $2\frac{d\theta}{dt}$, and add; thus $$4\frac{d}{dt}\left\{\frac{d}{dt}(\sin\theta\sin\psi)\right\}^2 + \frac{d}{dt}\left\{\frac{d}{dt}(\sin\theta\cos\psi)\right\}^2$$
$$+4\frac{d}{dt}\left\{\frac{d}{dt}(\cos\theta)\right\}^2 + 6\frac{g}{a}\frac{d}{dt}(\cos\theta) = 0,$$
$$\therefore \quad 4\left\{\frac{d}{dt}(\sin\theta\sin\psi)\right\}^2 + \left\{\frac{d}{dt}(\sin\theta\cos\psi)\right\}^2 + 4\left\{\frac{d}{dt}(\cos\theta)\right\}^2$$
$$= 6\frac{g}{a}(\cos\theta' - \cos\theta);$$
θ' being the initial value of θ.

This result can be immediately obtained by the principle of *Vis Viva*. Thus, $$\int dr\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \text{const.} - 2g\int dr\, z.$$
The conclusion is evident.

4567. (By W. Siverly.)—A particle moves from rest down the convex side of the Cissoid of Diocles, which is so placed as to have its asymptote vertical, the motion beginning at the cusp; show that the particle will leave the curve when it has moved a horizontal distance equal to eight-ninths of the radius of the generating circle.

Solution by the Rev. J. R. Wilson, M.A.; R. Tucker, M.A.; *and others.*

Resolving along the normal at the point where the particle leaves the curve, since the pressure at that point vanishes, we have $\frac{v^2}{\rho} = g\frac{dx}{ds}$.

But $v^2 = 2gy$, hence $\qquad \frac{dx}{s} = 2y, \text{ or } 1+\left(\frac{dy}{dx}\right)^2 = 2y\frac{d^2y}{dx^2}$ .... (1),

a result that is true for any plane curve.

In the particular case of the cissoid, $y^2 = \frac{x^3}{2a-x}$, whence we obtain

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}}(3a-x)}{(2a-x)^{\frac{3}{2}}}, \quad \frac{d^2y}{dx^2} = \frac{3a^2}{x^{\frac{1}{2}}(2a-x)^{\frac{5}{2}}}.$$

Substituting these values in (1), we obtain

$$\frac{8a^3-3a^2x}{(2a-x)^3} = \frac{6a^2x}{(2a-x)^3},$$

and therefore $x = \frac{8}{9}a$.

4774. (By Prof. Townsend, F.R.S.)—Any three conjugate diameters of an ellipsoid being supposed to meet the director sphere of the surface; show that the plane determined by any three points of meeting of different diameters touches the ellipsoid.

I. *Solution by* J. J. Walker, M.A.

Let the plane so determined be $Px+Qy+Rz = S$, the equation to the ellipsoid being $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$; then, the cone having its vertex at the centre and the section of the director sphere by this plane as base, will be

$$(a^2+b^2+c^2)(Px+Qy+Rz)^2 = S^2(x^2+y^2+z^2),$$

or $\qquad Ax^2+By^2+Cz^2+2(a^2+b^2+c^2)(QRyz+RPzx+PQxy) = 0,$

where $\qquad A = (a^2+b^2+c^2)P^2-S^2, \quad B = (a^2+b^2+c^2)Q^2-S^2,$

$$C = (a^2+b^2+c^2)R^2-S^2.$$

But when a right cone $Ax^2+By^2+Cz^2+ \ldots\ldots = 0$ contains three conjugate diameters of the ellipsoid, the relation $Aa^2+Bb^2+Cc^2 = 0$ obtains (Question 4801). This relation in the present case gives

$$(a^2+b^2+c^2)(P^2a^2+Q^2b^2+R^2c^2-S^2) = 0;$$

but the vanishing of the latter factor is the condition that the plane $Px+Qy+Rz = S$ shall touch the ellipsoid.

II. *Solution by* Prof. WOLSTENHOLME.

Refer the ellipsoid to the conjugate diameters as axes of coordinates, and let its equation be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; then the equations of the planes passing through the points where three diameters meet the director sphere are $\pm x \pm y \pm z = (a^2+b^2+c^2)^{\frac{1}{2}}$, which are tangent planes to the ellipsoid at the points $\pm\frac{x}{a^2}=\pm\frac{y}{b^2}=\pm\frac{z}{c^2}=\frac{1}{(a^2+b^2+c^2)^{\frac{1}{2}}}$.

The property also follows at once from the fact that parallelepipeds can be inscribed in the sphere whose faces touch the ellipsoid, for the diagonals of any parallelepiped circumscribing the ellipsoid are conjugate diameters. Thus the property is true if instead of the director sphere we take any conicoid whose axes $(2a', 2b', 2c')$ are coincident in direction with the axes $(2a, 2b, 2c)$ of the given ellipsoid, provided that $\frac{a^2}{a'^2}+\frac{b^2}{b'^2}+\frac{c^2}{c'^2}=1$, the relation between the invariants being (I believe)

$$4\Phi\Theta^2\Delta = \Theta^4 + 8\Delta^2\Theta\Theta' + 16\Delta^3\Delta'.$$

I may notice here, that for some purposes the analogue in three dimensions of the director circle in two dimensions is not the director sphere, but the locus of the intersection of three tangent lines mutually at right angles; thus, for the point spheres which have double contact with the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the locus of the points of contact is the curve of intersection of the ellipsoid with the ellipsoid $\frac{x^2}{a^2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+\ldots=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$, the locus aforesaid. (Of course these points are impossible, but such theorems may be usefully generalized.) It is simpler to give it as the intersection with the cone $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=0$. A theorem derived from this by a generalization equivalent to projection in two dimensions is: Given a conicord S and a plane conic U, if we draw a series of conicoids through U and having double contact with S, there will be three systems of such conicoids, the chords of contact of which pass through three fixed points in the plane of U (forming a self-conjugate triad to U, and with the pole of U a tetrahedron self-conjugate to S), and the points of contact will all lie on a quadric cone whose vertex is the pole of the plane of U with respect to S; also the locus of points from which can be drawn three tangent lines to S, meeting the plane of U in the corners of a self-conjugate triangle, will be a conicoid meeting S in the points of contact aforesaid.

III. *Solution by the* PROPOSER.

Denoting by O the common centre of the ellipsoid and sphere, by a, b, c the lengths of the three conjugate semi-diameters of the former, by p the length of the perpendicular from O on the plane determined by any three of their points of meeting with the latter, by r the radius of the sphere, and by θ the common value of the three equal angles α, β, γ made by p with a, b, c; then, since

$$p^2 = r^2\cos^2\theta = (a^2+b^2+c^2)\cos^2\theta = a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma,$$

therefore, &c.

IV. *Solution by* R. F. Davis, B.A.

Let $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2}=1$ be the equation to the ellipsoid referred to its conjugate diameters; and $x+y+z=\rho$ the equation to the plane, where $\rho=a^2+b^2+c^2=\alpha^2+\beta^2+\gamma^2$. Then the equation

$$(\alpha^2+\beta^2+\gamma^2)\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}\right)=(x+y+z)^2$$

represents the cone generated by a straight line through the origin which passes successively through every point of the curve of intersection of the plane with the ellipsoid. But this equation may be written in the form

$$\left(\frac{\gamma}{\beta}y-\frac{\beta}{\gamma}z\right)^2+\ldots+\ldots=0;$$

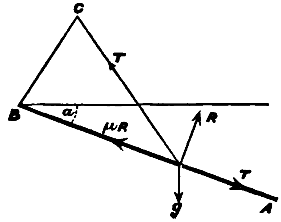
so that the cone degenerates into a straight line, and therefore the plane touches the ellipsoid.

4464. (By J. R. Wilson, M.A.)—AB is a fixed rough rod of length a, inclined to the horizon at an angle α, and carrying a very small ring. One end of an elastic string, whose natural length is a, is fastened to the higher end of the rod A, and passing through the ring is attached at the other end to a point C higher than B, such that ABC is a right angle, and BC $=c$. The weight of the ring is such that, when suspended freely from one end of the string, it would stretch it to a length AC. The ring is held at B, and then set free. Show (1) that if μ be the coefficient of friction, the velocity (v) of the ring at any point whose distance from B is x, is given by

$$v^2\left\{(a^2+c^2)^{\frac{1}{2}}-a\right\}=2g\left\{(x^2+c^2)\ -x\right\}(x+\mu c)+2(\mu\cos\alpha-\sin\alpha)gx-2\mu c^2g;$$

and (2) find the value of α, so that the ring may just reach the end A.

Solution by C. Leudesdorf, M.A.; *the* Proposer; *and others.*



1. Let T be the tension of the string; P the position of the rod at a time t; BP $=x$; then

$$a-x+(c^2-x^2)^{\frac{1}{2}}=a\left(1+\frac{T}{\lambda}\right),$$

and $\qquad (a^2+c^2)^{\frac{1}{2}}=a\left(1+\dfrac{ng}{\lambda}\right),$

taking the mass of the ring as unity.

Hence $\quad T=g\dfrac{(c^2+x^2)^{\frac{1}{2}}-x}{(c^2+a^2)^{\frac{1}{2}}-a}.$

Resolving along BA and a perpendicular to it, we have

$$\frac{d^2x}{dt^2} = \mu R + T\left(1 - \frac{x}{(c^2+x^2)^{\frac{1}{2}}}\right) - g \sin \alpha,$$

$$0 = R - g\cos\alpha + T\frac{c}{(c^2+x^2)^{\frac{1}{2}}};$$

therefore, $\dfrac{d^2x}{dt^2} = (\mu\cos\alpha - \sin\alpha) + \dfrac{(c^2+x^2)^{\frac{1}{2}} - x}{(c^2+a^2)^{\frac{1}{2}} - a}\left(1 - \dfrac{x+\mu c}{(x^2+c^2)^{\frac{1}{2}}}\right).$

Integrating, $\dfrac{v^2}{2g} = C + (\mu\cos\alpha - \sin\alpha)x + \dfrac{(x+\mu c)\left\{(c^2+x^2)^{\frac{1}{2}} - x\right\}}{(c^2+a^2)^{\frac{1}{2}} - a}.$

Since $v=0$ when $x=0$, therefore $C = -\dfrac{\mu c^2}{(c^2+a^2)^{\frac{1}{2}} - a}$,

and $v^2\left\{(c^2+a^2)^{\frac{1}{2}} - a\right\} = 2g\,(\mu\cos\alpha - \sin\alpha)\left\{(c^2+a^2)^{\frac{1}{2}} - a\right\}x$

$$+\,2g\,(x+\mu c)\left\{(c^2+x^2)^{\frac{1}{2}} - x\right\} - 2g\mu c^2.$$

In order that the ring may just reach A, v must vanish when $x = a$; therefore $(\mu\cos\alpha - \sin\alpha)\left\{(c^2+a^2)^{\frac{1}{2}} - a\right\}a + (a+\mu c)\left\{(c^2+a^2)^{\frac{1}{2}} - a\right\} - \mu c^2 = 0.$

2. Hence, putting $\mu = \tan\beta$, we find

$$\alpha = \beta + \sin^{-1}\left\{\frac{a\cos\beta + c\sin\beta}{a} - \frac{c^2\sin\beta}{a\left[(c^2+a^2)^{\frac{1}{2}} - a\right]}\right\}.$$

4692. (By J. L. McKenzie.)—Prove that (1) there are eight chords in an ellipse, each of which is normal to the curve at one extremity, and perpendicular to the central radius vector at the other, besides the line at infinity which, taken eight times, satisfies the above conditions; (2) these chords are all real and different, coincident in pairs, or all imaginary, according as $a : b$ is greater than, equal to, or less than $1+\sqrt{2}$; (3) if NR and N′R′ be two of these chords normal at points N and N′ in the same quadrant, show that $NR^2 + N'R'^2 = a^2 + b^2$; (4) the product of the tangents of the angles subtended at the centre by NP and N′P′ = 2; (5) if a chord NQ be drawn parallel to N′P′, the circle of curvature at N passes through Q; (6) the rectangle contained by the intercepts made by NP and N′P′ on the major axis, is equal to the rectangle contained by the intercepts on the minor axis and is constant for confocal ellipses.

Solution by Professor Nash.

1. Let θ be the eccentric angle of the point whose normal satisfies the given condition, and ϕ that of the other end of the normal, then

$$\frac{a^2\cos\phi}{\cos\theta}-\frac{b^2\sin\phi}{\sin\theta}=a^2-b^2,\quad\text{and}\quad \cos\phi\cos\theta+\sin\phi\sin\theta=0.$$

Eliminating ϕ, $\quad a^4\tan^4\theta-(a^4-4a^2b^2+b^4)\tan^2\theta+b^4=0$ .... (A).
This equation gives four values of $\tan\theta$, and therefore eight values of θ.

2. These values will be real and different, equal in pairs, or all imaginary, according as $(a^4-4a^2b^2+b^4)^2 > = < 4a^4b^4$,
that is, according as $\quad (a^4-6a^2b^2+b^4)(a^2-b^2)^2 > = < 0$,

or as $\qquad \dfrac{a^2}{b^2} > = < 3+2\sqrt{2}$, or as $\dfrac{a}{b} > = < 1+\sqrt{2}$.

3. From (A), $\qquad a^2\tan^2\theta+b^2=\pm(a^2-b^2)\tan\theta$ .... (B),
and for each sign we get a pair of points in the same quadrant; also

$$NR^2=a^2(\cos\theta_1-\cos\phi)^2+b^2(\sin\theta_1-\sin\phi)^2.$$

But, by the given conditions, $\cos\phi=-\sin\theta_1$, $\sin\phi=\cos\theta_1$,
therefore $NR^2=a^2+b^2+(a^2-b^2)\sin 2\theta_1$, $N'R'^2=a^2+b^2+(a^2-b^2)\sin 2\theta_2$,
Tan θ_1, $\tan\theta_2$ being roots of (B), and taking the lower sign,

$$\begin{aligned}\therefore\ NR^2+N'R'^2 &= 2(a^2+b^2)+(a^2-b^2)(\sin 2\theta_1+\sin 2\theta_2)\\ &= 2(a^2+b^2)+2(a^2-b^2)\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)\\ &= 2(a^2+b^2)+2(a^2-b^2)\frac{(\tan\theta_1+\tan\theta_2)(1+\tan\theta_1\tan\theta_2)}{(1+\tan^2\theta_1)(1+\tan^2\theta_2)}\\ &= 2(a^2+b^2)-2(a^2+b^2)\frac{(a^2-b^2)(a^2+b^2)}{2(a^2-b^2)^2}=a^2+b^2.\end{aligned}$$

4. The tangent of the angle between the lines

$$\frac{x}{a\cos\theta_1}=\frac{y}{b\sin\theta_1},\quad \frac{x}{a\sin\theta_1}+\frac{y}{b\cos\theta_1}=0,$$

is $$\frac{ab}{a^2-b^2}(\tan\theta_1+\cot\theta_1)=\frac{ab}{a^2-b^2}\,\frac{\tan^2\theta_1+1}{\tan\theta_1};$$

therefore product of two tangents

$$=\frac{a^2b^2}{(a^2-b^2)^2}\cdot\frac{2(a^2-b^2)^2}{a^2b^2}=2.$$

5. The coordinates of the point where the circle of curvature at θ_1 meets the ellipse are $a\cos 3\theta_1$, $-b\sin 3\theta_1$, and the equation of the common chord is

$$\frac{x-a\cos\theta_1}{a(\cos 3\theta_1-\cos\theta_1)}=-\frac{y-b\sin\theta_1}{b(\sin 3\theta_1+\sin\theta_1)},$$

or $$\frac{x-a\cos\theta_1}{a\sin\theta_1}=\frac{y-b\sin\theta_1}{b\cos\theta_1},$$

which is parallel to the normal at n', since

$$\tan\theta_1 \tan\theta_2 = \frac{b^2}{a^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(C)}.$$

6. The rectangles of the intercepts on the axes respectively are

$$\frac{(a^2-b^2)^2}{a^2}\cos\theta_1\cos\theta_2, \quad \frac{(a^2-b^2)^2}{b^2}\sin\theta_1\sin\theta_2,$$

and, by (C), these are equal. Also each of them is equal to

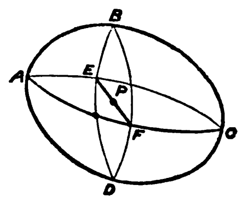
$$\left(\frac{a^2-b^2}{a}\right)^2 \frac{1}{(1+\tan^2\theta_1)^{\frac{1}{2}}(1+\tan^2\theta_2)^{\frac{1}{2}}} = \frac{(a^2-b^2)^2}{a^2}\,\frac{a^2}{(a^2-b^2)\sqrt{2}} = \frac{a^2-b^2}{\sqrt{2}},$$

and this is constant for confocal ellipses.

4807. (By H. Hart.)—Through a given point P on a sphere, draw an arc of a great circle which shall have its extremities on two given great circles and shall be bisected in P.

Solution by the Proposer.

Let the great circle whose pole is P cut the given great circles AFC, BED, in AC and B, D respectively. Draw the great circles AEC, BFD, making the same angles with APC, BPD that AFC, BED do, then the arc EF passes through and is bisected by P.



4798. (By the Rev. F. D. Thomson, M.A.)—A curve of the nth class is drawn touching the n^2 lines that join the vertices of two polygons of n sides $abc\ldots$, $a'b'c'\ldots$ Show that, if p and q be any two of the foci,

$$\frac{pa\,.\,pb\,.\,pc\ldots}{qa\,.\,qb\,.\,qc\ldots} = \frac{pa'\,.\,pb'\,.\,pc'\ldots}{qa'\,.\,qb'\,.\,qc'\ldots}$$

I. *Solution by* E. B. Elliott, B.A.

Using that system of line-coordinates in which the perpendicular distances from three fixed points of reference on any straight line are taken as the coordinates of that line, let $\alpha=0$, $\beta=0$, $\gamma=0$ ......, $\alpha'=0$, $\beta'=0$,

$\gamma' = 0$ ...... be the equations of the points a, b, c ......, a', b', c' ...... respectively. Then the equation of the curve is of the form

$$\alpha\beta\gamma \ldots\ldots = k\alpha'\beta'\gamma' \ldots\ldots;$$

for this and no other equation of the nth degree in the variables is satisfied by the coordinates of every line joining one vertex of the first set to one of the second.

Now the lines running from either focus p to the two circular points at infinity, are tangents to the curve. Hence, inserting in the above equation, the perpendiculars from the vertices on them are connected by the relations $\alpha_1\beta_1\gamma_1 \ldots\ldots = k\alpha_1'\beta_1'\gamma_1' \ldots\ldots$, and $\alpha_2\beta_2\gamma_2 \ldots\ldots = k\alpha_2'\beta_2'\gamma_2' \ldots\ldots$;

therefore $\quad \alpha_1\alpha_2 . \beta_1\beta_2 . \gamma_1\gamma_2 \ldots\ldots = k^2 \alpha_1'\alpha_2' . \beta_1'\beta_2' . \gamma_1'\gamma_2' \ldots\ldots;$

that is to say, $\quad \dfrac{pa^2}{1+i^2} . \dfrac{pb^2}{1+i^2} . \dfrac{pc^2}{1+i^2} \ldots\ldots = k^2 \dfrac{pa'^2}{1+i^2} . \dfrac{pb'^2}{1+i^2} . \dfrac{pc'^2}{1+i^2} \ldots\ldots;$

or, since the denominators on the two sides, though vanishing, are equal,

$$pa . pb . pc \ldots\ldots = k\, pa' . pb' . pc' \ldots\ldots$$

Similarly $\quad qa . qb . qc \ldots\ldots = k\, qa' . qb' . qc' \ldots\ldots$

Therefore $\quad \dfrac{pa . pb . pc \ldots}{qa . qb . qc \ldots} = \dfrac{pa' . pb' . pc' \ldots}{qa' . qb' . qc' \ldots}$

II. *Solution by the* PROPOSER.

The equation to the curve may be written

$$abc \ldots = \kappa\, a'b'c' \ldots$$

where $a, b, c \ldots$ are the equations to the points $a, b, c \ldots$,

or $\quad (x_1\lambda + y_1\mu + \nu)(x_2\lambda + \ldots) \ldots = \kappa(x'_1\lambda + y'_1\mu + \nu)(\ldots) \ldots\ldots\ldots$ (i.)

But the line pi, joining the locus p to the circular point i, is a tangent, and similarly for pj.

Hence the equation (i.) is satisfied by $\lambda : \mu : \nu = 1 : \pm\sqrt{-1} : 0$, or

$$(x_1 \pm y_1\sqrt{-1})(\ldots) = \kappa(x'_1 \pm y'_1\sqrt{-1})(\ldots) \ldots,$$

or $\quad (x_1^2 + y_1^2)(\ldots) = \kappa(x'^2_1 + y'^2_1)(\ldots) \ldots,$

or $\quad ap^2 . bp^2 \ldots = \kappa\, a'p^2 . b'^2p^2 \ldots$

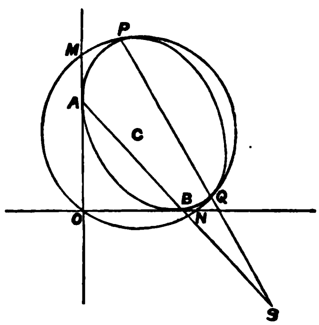
Similarly $\quad aq^2 . bq^2 \ldots = \kappa\, a'q^2 b'^2q^2 \ldots;$

therefore $\quad \dfrac{ap . bp \ldots}{aq . bq \ldots} = \dfrac{a'p . b'p \ldots}{a'q . b'q \ldots}.$

4606. (By J. L. McKenzie.)—A conic touches the rectangular coordinate axes at the points A and B, and a circle passing through the origin has double contact with the conic at the points P and Q. Prove (1) that the line joining the centre of the circle with the intersection of the

lines AB and PQ, passes through the points in which the circle cuts the axes; (2) that the line joining the centre of the circle with the origin is harmonically divided by the lines AB and PQ; and (3) hence deduce that if a rectangular hyperola has double contact with a parabola, the line joining the intersection of the common tangents with the centre of the hyperbola is bisected by the directrix of the parabola. (See Wolstenholme's *Problems*, No. 710.)

Solution by the PROPOSER.



1, 2. Any chord of a circle, which touches a concentric circle, is bisected at the point of contact. Hence, by projection, (Salmon's *Conics*, page 307,) if two conics have double contact with each other, any tangent to the one conic is harmonically divided by the other conic and the chord of contact. Therefore, if S be the intersection of AB and PQ, and M and N the points in which OA and OB cut the circle again, then SO, SB, SN, and SQ form a harmonic pencil; also SO, SA, SM, SP form a harmonic pencil; therefore S, N, and M are in a straight line, which evidently passes through the centre of the circle.

3. Reciprocating with regard to a circle whose centre is O, the polar reciprocal of the conic is a rectangular hyperbola, and that of the circle a parabola having double contact with the hyperbola. To the point S will correspond the line s which joins the centre of the hyperbola with the pole of the chord of contact; to the harmonic pencil SO, SA, SC, SP will correspond four harmonic points on the line s, viz., the point at ∞ on s, the centre of the hyperbola, the intersection of s with the directrix of the parabola, and the pole of the chord of contact. And since one of these points is at ∞, its harmonic conjugate bisects the distance between the other two.

4794. (By the EDITOR.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \cos(\alpha\tan\theta)\left(\epsilon^{\beta\tan\theta}+\epsilon^{-\beta\tan\theta}\right)d\theta = \pi\epsilon^{-\alpha}\cos\beta \;\ldots\ldots\ldots\; (1),$$

$$\int_0^{\frac{1}{2}\pi} \sin(\alpha\tan\theta)\left(\epsilon^{\beta\tan\theta}-\epsilon^{-\beta\tan\theta}\right)d\theta = \pi\epsilon^{-\alpha}\sin\beta \;\ldots\ldots\ldots\; (2).$$

Solution by Professor WOLSTENHOLME.

If in the well known integral $\int_0^\infty \frac{\cos ax}{1+x^2} = dx \frac{\pi}{2} \epsilon^{-a}$ (a positive) we put $a+i\beta$ for a, and *assume* that the equation holds, we get the two integrals

$$\int_0^\infty \frac{\cos ax\,(\epsilon^{\beta x}+\epsilon^{-\beta x})}{1+x^2}\,dx = \pi\epsilon^{-a}\cos\beta,$$

$$\int_0^\infty \frac{\sin ax\,(\epsilon^{\beta x}-\epsilon^{-\beta x})}{1+x^2}\,dx = \pi\epsilon^{-a}\sin\beta;$$

which are equivalent to the results given in the Question.

4787. (By H. W. HARRIS.)—Prove that

$$3\left\{\frac{\sin^2(\theta-\alpha)}{\sin^2(\alpha-\beta)\sin^2(\alpha-\gamma)}+\frac{\sin^2(\theta-\beta)}{\sin^2(\beta-\alpha)\sin^2(\beta-\gamma)}+\frac{\sin^2(\theta-\gamma)}{\sin^3(\gamma-\alpha)\sin^2(\gamma-\beta)}\right\}$$

$$\times\left\{\frac{\sin^5(\theta-\alpha)}{\sin^5(\alpha-\beta)\sin^5(\alpha-\gamma)}+\frac{\sin^5(\theta-\beta)}{\sin^5(\beta-\gamma)\sin^5(\beta-\alpha)}+\frac{\sin^5(\theta-\gamma)}{\sin^5(\gamma-\alpha)\sin^5(\gamma-\beta)}\right\}$$

$$=5\left\{\frac{\sin^3(\theta-\alpha)}{\sin^3(\alpha-\beta)\sin^3(\alpha-\gamma)}+\frac{\sin^3(\theta-\beta)}{\sin^3(\beta-\alpha)\sin^3(\beta-\gamma)}+\frac{\sin^3(\theta-\gamma)}{\sin^3(\gamma-\alpha)\sin^3(\gamma-\beta)}\right\}$$

$$\times\left\{\frac{\sin^4(\theta-\alpha)}{\sin^4(\alpha-\beta)\sin^4(\alpha-\gamma)}+\frac{\sin^4(\theta-\beta)}{\sin^4(\beta-\alpha)\sin^4(\beta-\gamma)}+\frac{\sin^4(\theta-\gamma)}{\sin^4(\gamma-\alpha)\sin^4(\gamma-\beta)}\right\}$$

$$=\frac{30}{7}\left\{\frac{\sin^7(\theta-\alpha)}{\sin^7(\alpha-\beta)\sin^7(\alpha-\gamma)}+\frac{\sin^7(\theta-\beta)}{\sin^7(\beta-\alpha)\sin^7(\beta-\gamma)}+\frac{\sin^7(\theta-\gamma)}{\sin^7(\gamma-\alpha)\sin^7(\gamma-\beta)}\right\}.$$

I. *Solution by* C. LEUDESDORF, M.A.

Let $$x \equiv \frac{\sin(\theta-\alpha)}{\sin(\gamma-\alpha)\sin(\alpha-\beta)},\quad y\equiv \text{\&c.}\quad z\equiv\text{\&c.};$$

then, since $x+y+z=0$, we may assume x, y, z to be the roots of the equation $x^3+p_2x+p_3=0$, and we have

$S_1=0,\quad S_2=-2p_2,\quad S_3=-3p_3,\quad S_4=2p_2^2,\quad S_5=5p_2p_3,\quad S_7=-7p_2^2p_3,$

whence $$3S_2S_5=-30p_2^2p_3=5S_3S_4=\frac{30}{7}S_7.$$

II. *Solution by* R. F. Davis, B.A.

If $p+q+r=0$, the relations subsisting between the different values of $S_n\left(\equiv\dfrac{p^n+q^n+r^n}{n}\right)$ may be found as follows.

Assume $$(1-px)(1-qx)(1-rx) = 1-ux^2-vx^3.$$

Take logarithms, expand, and equate coefficients of like powers of x. We shall thus obtain $3\,S_2S_5 = 5\,S_3S_4 = \frac{30}{7}\,S_7$. (See Wolstenholme's *Problems*, Ex. 113.)

By a suitable and legitimate substitution the formidable relations of the question may now be obtained. Put $p = \dfrac{\tan\theta-\tan\alpha}{(\tan\alpha-\tan\beta)(\tan\alpha-\tan\gamma)}$, with similar values for q, r.

4752. (By Professor Cayley.)—Mr. Wolstenholme's Question 3067 may evidently be stated as follows:—

If (a, b, c) are the coordinates of a point on the cubic curve
$$a^3+b^3+c^3 = (b+c)(c+a)(a+b),$$
and if $$(b^2+c^2-a^2)\,x = (c^2+a^2-b^2)\,y = (a^2+b^2-c^2)\,z;$$
then (x, y, z) are the coordinates of a point on the same cubic curve.

This being so, it is required to find the geometrical relation of the two points to each other.

Solution by Professor Cayley.

1. On referring to Professor Wolstenholme's Solution of the original Question 3067 (*Reprint*, Vol. XIII., p. 70), it appears that the coordinates (x, y, z) of the point in question may be expressed in the more simple form $$x : y : z = a(-a+b+c) : b(a-b+c) : c(a+b-c);$$
viz., the given relation between (a, b, c) being equivalent to
$$4abc+(-a+b+c)(a-b+c)(a+b-c) = 0,$$
we have $$a^2-(b-c)^2 = \frac{-4abc}{-a+b+c},$$
and thence $$b^2+c^2-a^2 = 2bc\left(1+\frac{2a}{-a+b+c}\right) = 2bc\left(\frac{a+b+c}{-a+b+c}\right);$$
and consequently $$(b^2+c^2-a^2)\,x = \frac{2abc\,(a+b+c)\,x}{a\,(-a+b+c)},$$
whence the transformation in question.

2. Writing for greater symmetry (x, y, z) in place of (a, b, c), and

(x', y', z') in place of (x, y, z), the coordinates (x, y, z) and (x', y', z') of the two points are connected by the relation

$$x' : y' : z' = x(-x+y+z) : y(x-y+z) : z(x+y-z),$$

and we thence at once deduce the converse relation

$$x : y : z = x'(-x'+y'+z') : y'(x'-y'+z') : z'(x'+y'-z').$$

Hence, writing $(-x+y+z,\ x-y+z,\ x+y-z) = (\xi, \eta, \zeta)$,

and similarly $(-x'+y'+z',\ x'-y'+z',\ x'+y'-z') = (\xi', \eta', \zeta')$,

we have $x' : y' : z' = x\xi : y\eta : z\zeta,\quad x : y : z = x'\xi' : y'\eta' : z'\zeta'$,

and thence also $\xi\xi' = \eta\eta' = \zeta\zeta'$; so that, regarding (ξ, η, ζ), (ξ', η', ζ') as the coordinates of the two points, we see that these are inverse points one of the other in regard to the triangle $\xi=0$, $\eta=0$, $\zeta=0$.

To complete the solution, we must introduce these new coordinates into the equation of the cubic curve. Writing this under the form

$$8xyz + 2(-x+y+z)(x-y+z)(x+y-z) = 0,$$

and observing that $(2x, 2y, 2z) = (\eta+\zeta, \zeta+\xi, \xi+\eta)$,

the equation is $(\eta+\zeta)(\zeta+\xi)(\xi+\eta) + 2\xi\eta\zeta = 0$;

viz., this is a cubic curve inverting into itself. And the two points in question are thus any two inverse points on this cubic curve.

3. In regard to the original form, that the point (x, y, z) defined by the equations $x(-a^2+b^2+c^2) = y(a^2-b^2+c^2) = z(a^2+b^2-c^2)$,

lies on the cubic curve $a^3+b^3+c^3-(b+c)(c+a)(a+b) = 0$,

Professor Sylvester proceeds as follows:—Writing

$$(x, y, z) = \left\{a^4-(b^2-c^2)^2,\ b^4-(c^2-a^2)^2,\ c^4-(a^2-b^2)^2\right\},\ = (\mathrm{A}, \mathrm{B}, \mathrm{C}) \text{ suppose};$$

and $\mathrm{F}(a, b, c) = a^3+b^3+c^3-(b+c)(c+a)(a+b)$,

he observes that the truth of the theorem depends on the identity

$$\mathrm{F}(\mathrm{A}, \mathrm{B}, \mathrm{C}) + \mathrm{F}(a, b, c)\,\mathrm{F}(a, -b, c)\,\mathrm{F}(a, b, -c)\,\mathrm{F}(a, -b, -c) = 0,$$

and that, in order to prove the identity generally, it is sufficient to prove it for the three cases $a^2=0$, $a^2 = b^2+c^2$, $a^2=b^2$, which may be effected without difficulty.

4. But, for a general proof of the identity, write $\lambda = b^2+c^2$, $\mu = b^2-c^2$, so that $\mathrm{A} = a^4-\mu^2$, $\mathrm{B} = (a^2+\mu)(-a^2+\lambda)$, $\mathrm{C} = (-a^2+\lambda)(a^2-\mu)$, whence

$$\begin{aligned}-\mathrm{F}(\mathrm{A}, \mathrm{B}, \mathrm{C}) &= -(a^4-\mu^2)^3 + 2(a^2-\lambda)^3(a^6+3a^2\mu^2) - 8a^2b^2c^2(a^4-\mu^2)(a^2-\lambda),\\ &= a^{12} - 6\lambda a^{10} + (6\lambda^2+9\mu^2-8b^2c^2)a^8 + \lambda(-2\lambda^2-18\mu^2+8b^2c^2)a^6\\ &\quad + \mu^2(18\lambda^2-3\mu^2+8b^2c^2)a^4 + \lambda\mu^2(-6\lambda^2-8b^2c^2)a^2 + \mu^6.\end{aligned}$$

Moreover $\mathrm{F}(a, b, c) = a\left\{a^2-(b+c)^2\right\} - (b+c)\left\{a^2-(b-c)^2\right\}$,

therefore $\mathrm{F}(a, -b, -c) = a\left\{a^2-(b+c)^2\right\} + (b+c)\left\{a^2-(b-c)^2\right\}$;

whence $\mathrm{F}(a,b,c)\,\mathrm{F}(a, -b, -c) = a^2\left\{a^2-(b+c)^2\right\}^2 - (b+c)^2\left\{a^2-(b-c)^2\right\}^2$,

which is $= a^6 - 3\gamma^2a^4 + \gamma^2(\gamma^2+2\delta^2)a^2 - \gamma^2\delta^2$,

if $\gamma = b + c$, $\delta = b - c$. By changing the sign of c, we interchange γ and δ, and we thus have

$$F(a, b, -c)\, F(a, -b, c) = a^6 - 3\delta^2 a^4 + \delta^2 (2\gamma^2 + \delta^2)\, a^2 - \gamma^4\delta^2,$$

and the identity to be verified is thus

$$\left\{a^6 - 3\gamma^2 a^4 + \gamma^2(\gamma^2 + 2\delta^2)\, a^2 - \gamma^2\delta^4\right\} \left\{a^6 - 3\delta^2 a^4 + \delta^2(2\gamma^2 + \delta^2)\, a^2 - \gamma^4\delta^2\right\}$$
$$= a^{12} - 6\lambda a^{10} \ldots\ldots + \mu^6, \text{ ut suprà;}$$

the values of λ, μ in terms of γ, δ are $\lambda = \frac{1}{2}(\gamma^2 + \delta^2)$, $\mu = \gamma\delta$; and substituting these values on the right-hand side the verification can be completed without difficulty.

4797. (By J. W. L. Glaisher, F.R.S.)—If Px denote the number of partitions of x into the even elements 2, 4, 6, ..., without repetitions, and Qx the number of partitions of x into the uneven elements 1, 3, 5... also without repetitions, prove that

$$Px + 2P(x-1) + 2P(x-4) + 2P(x-9) + \&c.$$
$$= Qx + Q(x-1) + Q(x-3) + Q(x-6) + \&c.,$$

1, 4, 9... being the squares, and 1, 3, 6... the triangular numbers.

Solution by the Proposer.

The theorem, in analytical language, is that

$$(1 + 2q + 2q^4 + 2q^9 + \ldots) \,.\, 1 + q^2 \,.\, 1 + q^4 \,.\, 1 + q^6 \ldots$$
$$= (1 + q + q^3 + q^6 + \ldots) \,.\, 1 + q \,.\, 1 + q^3 \,.\, 1 + q^5 \ldots,$$

an identity which is immediately derivable from the formulæ

$$\frac{1-q \,.\, 1-q^2 \,.\, 1-q^3 \ldots}{1+q \,.\, 1+q^2 \,.\, 1+q^3 \ldots} = 1 - 2q + 2q^4 - 2q^9 + \&c.,$$

$$\frac{1-q^2 \,.\, 1-q^4 \,.\, 1-q^6 \ldots}{1-q \,.\, 1-q^3 \,.\, 1-q^5 \ldots} = 1 + q + q^3 + q^6 + q^{10} + \&c.$$

(*Fundamenta Nova*, p. 185.)

4838. (By B. Williamson, M.A.)—At any point on a curve of the nth degree, show that the radii of curvature of the series of polar curves, taken with respect to the point, are in harmonic progression.

Solution by E. B. Elliott, B.A.

Taking the point for origin, and the normal and tangent at it for axes, the equation is of the form

$$y = ax^2 + 2hxy + by^2 + u_3 + u_4 + \ldots\ldots + u_n.$$

The successive polar curves of the origin with respect to this are

$$(n-1)z = (n-2)(ax^2+2hxy+by^2) + (n-3)u_3 + (n-4)u_4 + \ldots\ldots + u_{n-1},$$

$$(n-1)(n-2)z = (n-2)(n-3)(ax^2+2hxy+by^2) + (n-3)(n-4)u_3 + \ldots\ldots,$$

$$(n-1)(n-2)(n-3)z = (n-2)(n-3)(n-4)(ax^2+2hxy+by^2) + (n-3)(n-4)(n-5)u_3 + \ldots\ldots,$$

and so on.

Thus the successive radii of curvature at the origin are, beginning with that of the curve itself,

$$\frac{1}{2a},\ \frac{n-1}{n-2}\cdot\frac{1}{2a},\ \frac{n-1}{n-3}\cdot\frac{1}{2a},\ \ldots\ldots,$$

which are proportional to $\quad \dfrac{1}{n-1},\ \dfrac{1}{n-2},\ \dfrac{1}{n-3},\ \ldots\ldots,$

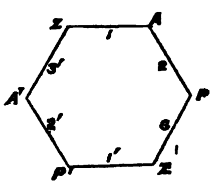
and so form an harmonic series.

4741. (By T. Cotterill, M.A.)—If the opposite sides of the hexagon ZAPZ′P′A′ be parallel, then the triangles ZPP′ and Z′A′A are equal.

Hence, or otherwise, if A, A′ are fixed points and P, P′ variable points collinear with the fixed point Z, and such that AP and A′P′ are parallel, then the parallel through P′ to ZA, and the parallel through P to ZA′, meet on the line AA′; and consequently, if K be any point on AA′, the distance of P′ from the parallel to ZA through K is to the distance of P from the parallel to ZA′ through K in the ratio of ZA′ to ZA.

If one of the points P, P′ describe a curve, order m, the other describes a curve, order $2m$, passing m times through three fixed points.

Solution by the Rev. F. D. Thomson, M.A.



1. Call the sides in order, beginning with ZA, 1, 2, 3, 1′, 2′, 3′; and let their equations be

$$x=0,\quad y=0,\quad \frac{x}{a}+\frac{y}{b}-1=0 \ \ldots\ldots\ (1, 2, 3),$$

$$x-h=0,\quad y-k=0,\quad \frac{x}{a}+\frac{y}{b}-m=0 \ldots (1', 2', 3').$$

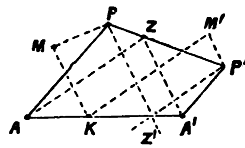
Then $2\,ZPP' = \begin{vmatrix} 0, & bm, & 1 \\ a, & 0, & 1 \\ h, & k, & 1 \end{vmatrix} \sin^2 A = [bm(h-a) + ak]\sin^2 A,$

$$2\,ZAA' = \begin{vmatrix} h, & b\left(1-\frac{h}{a}\right), & 1 \\ 0, & 0, & 1 \\ \left(m-\frac{k}{b}\right), & k, & 1 \end{vmatrix} \sin^2 A$$

$$= \left[-hk + ab\left(1-\frac{h}{a}\right)\left(m-\frac{k}{b}\right)\right] \sin^2 A$$

$$= -\left[bm\,(h-a) + ak\right] \sin^2 A,$$

therefore, neglecting sign, $ZPP' = ZAA'$.



2. Hence, or more simply by Pascal's Theorem, with the construction of the question, $AZ'A'$ is a straight line. Let PM, P'M' be the distances of the parallels to Z P, Z P', through Z', from P and P'. Then

$$PM : P'M' = \sin KZ'P : \sin Z'KM'$$
$$= \sin AA'Z : \sin A'AZ = AZ : A'Z.$$

3. The order of the curve traced out by P' will be found by seeing how many points of the locus are on the line at infinity. Now, when P is any point on AZ, P' lies at infinity in the direction of AZ; hence, if the locus of P be of order m, there are m points at infinity in the direction AZ. Again, to any point at infinity in the locus of P corresponds a point at infinity in the same direction in the locus of P'. Hence there are m more points at infinity in the locus of P', and the line at infinity meets the locus in $2m$ points. Again, to each point on A'Z corresponds the point A', and to each point on a parallel to A'Z through A corresponds the point Z. Hence the curve passes m times through A', Z, and the point at infinity in the direction AZ.

4805. (By R. W. Genese, M.A.)—ABCD is a quadrilateral inscribed in a conic, and F, G the intersections of pairs of opposite sides. Prove that every conic through ACFG will have with the given conic a chord of intersection that will always pass through a fixed point, viz., the pole of BD.

I. *Solution by* Professor Townsend, F.R.S.

It is a well known property (see Salmon's *Conic Sections*, Edit. 5, Art. 266) that the free chord of intersection of a variable conic, passing through four fixed points, with any fixed conic passing through two of the points, turns round a fixed point on the line connecting the remaining two points; hence, in the question, the free chord of intersection XY of the variable conic through A, C, F, G, with the fixed conic through A, C, B, D, passes through a fixed point Z on the line FG, and, the two tangents to the fixed conic at B and D being evidently two particular positions of XY, therefore, &c.

II. *Solution by* R. F. DAVIS, B.A., E. RUTTER, *and others*.

The following proposition is given in Salmon. Given four points on a conic section, its chord of intersection with a fixed conic passing through two of these points will pass through a fixed point. To find the position of this fixed point for the question proposed, we may proceed thus. Let the variable conic break up into the two straight lines AG, CF. Then one chord of intersection with the original conic is AC, and the other is the tangent at B. Thus the tangent at B passes through the fixed point; similarly, &c.

4778. (By the EDITOR.)—If $(x^2+y^2-a^2)^2 = 4a^2\left\{(a-x)^2+y^2\right\}$, prove that $\quad \left\{(a-x)^2+y^2\right\}^3 = \left\{(a-x)^3+(3a-x)y^2\right\}^2.$

I. *Solution by* Professor WOLSTENHOLME, M.A.

The second equation $\left\{(a-x)^2+y^2\right\}^3 = \left\{(a-x)^3+(3a-x)y^2\right\}^2$ becomes, on putting $a-x$ for x,

$$(x^2+y^2)^3 = \left\{x^3+(2a+x)y^2\right\}^2 = x^2(x^2+y^2)^2+4axy^2(x^2+y^2)+4a^2y^4;$$

or $$y^2(x^2+y^2)^2 = 4axy^2(x^2+y^2)+4a^2y^4;$$

or, suppressing the factor y^2, $(x^2+y^2-2ax)^2 = 4a^2(x^2+y^2)$;

or, restoring $a-x$ for x, $(x^2+y^2-a^2)^2 = 4a^2\left\{(a-x)^2+y^2\right\}$. Hence, when the latter equation is satisfied, so also is the former. (It is the equation of a Cardioide, whose cusp is at a, 0. A Cardioide* is the limit of a Cartesian, when its three single foci coincide; the equation $(x^2+y^2-\beta\gamma-\gamma\alpha-\alpha\beta)^2 + 4\alpha\beta\gamma(2x-\alpha-\beta-\gamma) = 0$, which is the equation of a Cartesian, referred to its triple focus as origin, becoming $(x^2+y^2-3a^2)^2+4a^3(2x-3a) = 0$, when $\beta=\gamma=\alpha$; and this is identical with

$$(x^2+y^2-a^2)^2 = 4a^2\left\{(x-a)^2+y^2\right\}.$$

If r, r_1, r_2, r_3, be the focal distances, we get, from this general equation,

$$r_1 = (\beta\gamma)^{\frac{1}{2}}+\left\{a(a+\beta+\gamma-2x)\right\}^{\frac{1}{2}} = \frac{r^2+\beta\gamma-\gamma\alpha-\alpha\beta}{2(\beta\gamma)^{\frac{1}{2}}}, \text{ \&c.},$$

whence we obtain at once the well-known focal equation of a Cartesian (given in the *Reprint*, Vol. XI., p. 56),

$$\alpha^{\frac{1}{2}}(\beta-\gamma)r_1+\beta^{\frac{1}{2}}(\gamma-\alpha)r_2+\gamma^{\frac{1}{2}}(\alpha-\beta)r_3 = 0.$$

* This is probably well known, but is not mentioned in some places where I should have expected to find it.

[The deduction of the values of r_1, r_2, r_3, from the general equation may be done as follows:—

$$(x^2+y^2-\beta\gamma-\gamma\alpha-\alpha\beta)^2+4\alpha\beta\gamma\,(2x-\alpha-\beta-\gamma)=0, \text{ and } r_1^2=(x-\alpha)^2+y^2;$$

therefore $\left\{r_1^2+\alpha\,(2x-\alpha-\beta-\gamma)-\beta\gamma\right\}^2+4\alpha\beta\gamma\,(2x-\alpha-\beta-\gamma)=0$,

or
$$\left\{r_1^2+\alpha\,(2x-\alpha-\beta-\gamma)+\beta\gamma\right\}^2=4\beta\gamma\,r_1^2;$$

therefore
$$(r_1-\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}})^2=\alpha\,(\alpha+\beta+\gamma-2x)=\frac{(r^2-\beta\gamma-\gamma\alpha-\alpha\beta)^2}{4\beta\gamma};$$

or
$$r_1=(\beta\gamma)^{\frac{1}{2}}+\frac{r^2-\beta\gamma-\gamma\alpha-\alpha\beta}{2\,(\beta\gamma)^{\frac{1}{2}}}, \text{ \&c. \&c.}$$

If it is attempted to get at the Cartesian from this last equation, the vanishing factors $(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha-\beta)$ must be got rid of first.]

II. *Solution by* R. F. Davis, B.A.

Let A be the point $(a, 0)$; P the point (x, y); and R the other point of intersection of AP with the circle $x^2+y^2=a^2$. From the given relation, $\text{PR}=2a$; so that the locus of P is the cardioid $r=2a\,(1+\cos\theta)$, A being the pole. Whence $r\,(1-\cos\theta)=2a\sin^2\theta$, and

$$r^3=r^3\cos^3\theta+(r\cos\theta+2a)\,r^2\sin^2\theta.$$

Then transform back to rectangular coordinates by putting

$$r\cos\theta=a-x, \text{ and } r\sin\theta=y.$$

III. *Solution by* C. Leudesdorf, M.A.

If $x^2+y^2-a^2\equiv p$, $(a-x)^2+y^2\equiv q$, $(a-x)^3+(3a-x)\,y^2\equiv r$;

we have
$$r-2aq=(a-x)\,(x^2+y^2-a^2)=\frac{p\,(q-p)}{2a};$$

therefore
$$2ar=4a^2q-p^2+pq.$$

Therefore, if $p^2=4a^2q$, $r=\dfrac{pq}{2a}$ and $r^2=\dfrac{p^2q^2}{4a^2}=q^3$.

IV. *Solution by* J. Hammond, E. Rutter, *and others.*

Since $x^2-a^2=2(a-x)^2-(a-x)\,(3a-x)$ and $4a^2=\left\{(3a-x)-(a-x)\right\}^2$, therefore the equation $(x^2+y^2-a^2)^2=4a^2\left\{(a-x)^2+y^2\right\}$ becomes

$$\begin{aligned}&4\,(a-x)^4-4\,(a-x)^3\,(3a-x)+(a-x)^2\,(3a-x)^2\\&\qquad\qquad+4y^2(a-x)^2-2y^2\,(a-x)\,(3a-x)+y^4\\&=(a-x)^4-2\,(a-x)^3\,(3a-x)+(a-x)^2\,(3a-x)^2\\&\qquad\qquad+y^2\,(a-x)^2-2y^2\,(a-x)\,(3a-x)+y^2\,(3a-x)^2,\end{aligned}$$

or
$$3\,(a-x)^4+3y^2\,(a-x)^2+y^4=2\,(a-x)^3\,(3a-x)+y^2\,(3a-x)^2.$$

Multiplying by y^2 and adding $(a-x)^6$ to each side, we have

$$\left\{(a-x)^2+y^2\right\}^3=\left\{(a-x)^3+(3a-x)\,y^2\right\}^2.$$

4785. (By H. Fortey.)—A pack of cards consists of l sets, each set containing p cards marked 1, 2, 3......p. All the sets are shuffled together and then dealt out face upwards on a table, the dealer calling as he does so 1, 2, 3......p, 1, 2, &c., until the pack is exhausted. For each card which corresponds with his call he scores one. Find (1) the chance of his making any particular score; also, as particular applications, find (2) the chance of scoring 7, if $l=3$, $p=3$; (3) the chance of scoring 4, if $l=3$, $p=4$; (4) the chance of scoring m when only r of the l rows are dealt out.

Solution by the Proposer.

1. Form a rectangle of lp compartments, the columns marked (1), (2), (3) ...... (p), and the the rows marked (1), (2), (3) ...... (l), thus:—

	(1),	(2),	(3)	.....	.....	.....	.....	(p)
(1)								
(2)								
⋮								
(l)								

and let the cards be dealt out in l rows, p cards in each row, so that there may be one card in each compartment. Let $lp=n$, then the number of possible arrangements is $\underline{|n}$.

The number of arrangements with a card marked *one* in the compartment (1, 1) is $l\,\underline{|n-1}$. Now there are l compartments in the column headed (1), and therefore it would appear that there are $l^2\,\underline{|n-1}$ arrangements with a card marked *one* in column (1).

But, counting in this way, every case in which *two ones* occur in the column has been counted twice over, and the number of these cases is $\dfrac{l^2(l-1)^2}{1.2}\,\underline{|n-2}$. Therefore, subtracting them once, the result becomes

$$l^2\,\underline{|n-1}-\frac{l^2(l-1)^2}{1.2}\,\underline{|n-2}.$$

But now the cases where *three ones* occur in the column have been added three times, and subtracted three times, and must therefore be added once. Proceeding in this way, we find ultimately that the number of arrangements with one or more *ones* in column (1) is

$$l^2\,\underline{|n-1}-\frac{l^2(l-1)^2}{1.2}\,\underline{|n-2}+\frac{l^2(l-1)^2(l-2)^2}{1.2.3}\,\underline{|n-3}-\text{\&c.},$$

when the series is continued to l terms, or (which is the same thing) until the terms vanish.

Therefore the number of arrangements in which *no one* is in column (1), is

$$\underline{|n}-l^2\,\underline{|n-1}+\frac{l^2(l-1)^2}{1.2}\,\underline{|n-2}-\text{\&c.}$$

Now if r cards marked *one, and no more*, are to lie in column (1), they can

be selected in $\dfrac{\lfloor l}{\lfloor l-r\,\lfloor r}$ ways, the compartments into which they are to be placed can be selected in the same number of ways, and each set of cards can be put in each set of compartments in $\lfloor r$ ways. Therefore the number of ways of arranging r cards marked *one* in column (1) is

$$\frac{(\lfloor l)^2}{(\lfloor l-r)^2\,\lfloor r}.$$

But there are $n-r$ cards remaining of which $l-r$ are marked *one*, and none of these last must be placed in any of the $l-r$ vacant compartments of column (1). Therefore the number of ways of arranging them is

$$\lfloor n-r-(l-r)^2\,\lfloor n-r-1+\frac{(l-r)^2\,(l-r-1)^2}{1\,.\,2}\,\lfloor n-r-2-\text{\&c.}$$

Therefore the number of arrangements, so that a score, or r exactly, shall be made in a specified column, is

$$\frac{(\lfloor l)^2}{(\lfloor l-r)^2\,\lfloor r}\left\{\lfloor n-r-(l-r)^2\,\lfloor n-r-1+\frac{(l-r)^2\,(l-r-1)^2}{1\,.\,2}\,\lfloor n-r-2-\text{\&c.}\right\}$$

Now let ϕ_r be a symbol of operation such that $\phi_r\,\lfloor n$ may be equal to the expression just written down. And let J be another symbol such that

$$\mathrm{J}\,\lfloor n=\lfloor n-1,\quad \mathrm{J}^2\,\lfloor n=\mathrm{J}\,\lfloor n-1=\lfloor n-2,\quad \text{\&c.}=\text{\&c.},$$

$$\mathrm{J}^r\,\lfloor n=\lfloor n-r,\quad \mathrm{J}^n\,\lfloor n=\lfloor 0=1.$$

Then

$$\phi_r\,\lfloor n=\frac{(\lfloor l)^2}{(\lfloor l-r)^2\,\lfloor r}\left\{\mathrm{J}^r\,\lfloor n-(l-r)^2\,\mathrm{J}^{r+1}\,\lfloor n+\frac{(l-r)^2\,(l-r-1)^2}{1\,.\,2}\,\mathrm{J}^{r+2}\,\lfloor n\ldots\right\}$$

$$=\frac{(\lfloor l)^2}{(\lfloor l-r)^2\,\lfloor r}\left\{1-(l-r)^2\,\mathrm{J}\,\lfloor n+\frac{(l-r)^2\,(l-r-1)^2}{1\,.\,2}\,\mathrm{J}^2\,\lfloor n-\text{\&c.}\right\}\mathrm{J}^r$$

or, separating symbols of operation and quantity,

$$\phi_r=\frac{(\lfloor l)^2}{(\lfloor l-r)^2\,\lfloor r}\left\{1-(l-r)^2\,\mathrm{J}+\frac{(l-r)^2\,(l-r-1)^2}{1\,.\,2}\,\mathrm{J}^2-\text{\&c.}\right\}\mathrm{J}^r.$$

If now we wish to find in how many ways we can score the numbers r, s, t, in three specified columns, and nothing in all the rest, then it is clear that we must operate on $\lfloor n$ with $\phi_0^{p-3}\,\phi_r\,\phi_s\,\phi_t$. But if the columns in which these particular scores are to be made are *not* specified, then we must operate on $\lfloor n$ with the term involving $\phi_0^{p-3}\,\phi_r\,\phi_s\,\phi_t$ in the expansion of $(\phi_0+\phi_1+\phi_2+\text{\&c.}+\phi_l)^p$, or $\dfrac{p\,(p-1)\,(p-2)}{1\,.\,2\,.\,3}\,\phi_0^{p-3}\,\phi_r\,\phi_s\,\phi_t$, which is evident, since the three columns can be selected in $\dfrac{p\,(p-1)\,(p-2)}{1\,.\,2\,.\,3}$ ways.

Finally, if a total score of m is to be made, but nothing is specified as to the columns in which the partial scores are to be made, or the way in which the total is to be made up, the number of arrangements will be got

by operating on $\lfloor n$ with the coefficient of x^m in the expansion of

$$(\phi_0 + \phi_1 x + \phi_2 x^2 + \text{\&c.} + \phi_l x^l)^p.$$

Therefore the *chance* of scoring m will be the result of that operation divided by $\lfloor n$.

2. Given $l = 3$, $p = 3$; to find the chance of scoring 7. Here we have to operate on $\lfloor 9$ with the coefficient of x^7 in the expansion of

$$(\phi_0 + \phi_1 x + \phi_2 x^2 + \phi_3 x^3)^3, \text{ or } 3\phi_3(\phi_1\phi_3 + \phi_2^2).$$

Now $\phi_1 = 9(1-4J+2J^2)J,\ \phi_2 = 18(1-J)J^2,\ \phi_3 = 6J^3;$

therefore
$$\begin{aligned} 3\phi_3(\phi_1\phi_3+\phi_2^2)\lfloor 9 &= 3 \,.\, 18^2(7J^7-16J^8+8J^9)\lfloor 9 \\ &= 3 \,.\, 18^2(7\lfloor 2 - 16\lfloor 1 + 8\lfloor 0) \\ &= 3 \,.\, 18^2(14-16+8) = 18^3, \end{aligned}$$

Therefore the chance of scoring $7 = \dfrac{18^3}{\lfloor 9} = \dfrac{9}{560}$.

3. If $l=3$, $p=4$; to find the chance of scoring 4. Here we must operate on $\lfloor 12$ with the coefficient of x^4 in the expansion of

$$(\phi_0+\phi_1 x + \phi_2 x^2 + \phi_3 x^3)^4, \text{ or } \phi_1^4 + 6\phi_0^2\phi_2^2 + 12\phi_0^2\phi_1\phi_3,$$

where ϕ_1, ϕ_2, ϕ_3, are the same as in (2), and $\phi_0 = 1-9J+18J^2-6J^3$.

The arithmetical result of the operation is 29, 035, 260, and therefore the chance of scoring 4 is this number divided by $\lfloor 12$, or $\dfrac{17923}{73920}$.

4. To find the chance of scoring m when only r of the l rows are dealt out. To find the number of arrangements we must operate on $\lfloor n$ with the coefficient of x^m in the expansion of

$$(\psi_0 + \psi_1 x + \psi_2 x^2 + \ldots\ldots + \psi_r x^r)^p,$$

where
$$\psi_s = \frac{\lfloor l \ \lfloor r}{\lfloor l-s \ \lfloor r-s \ \lfloor s}\left\{1-(l-s)(r-s)\,l + \frac{(l-s)(l-s-1)(r-s)(r-s-1)}{1\,.\,2}J^2 - \text{\&c.}\right\}J^s,$$

and the chance of the score will be got by dividing the result by $\lfloor n$.

5. In (4), if $r=1$, or only *one* row of cards is dealt out, the chance of scoring m becomes
$$\frac{\lfloor p}{\lfloor n \ \lfloor p-m \ \lfloor m}\,.\, l^m(1-lJ)^{p-m}\,.\,J^m\lfloor n.$$

If $m=0$, we have, for the chance of scoring 0,

$$\frac{1}{\lfloor n}(1-lJ)^p\lfloor n,$$

which is the result given on p. 156 of TODHUNTER'S *History of the Theory of Probability*, but more compactly expressed.

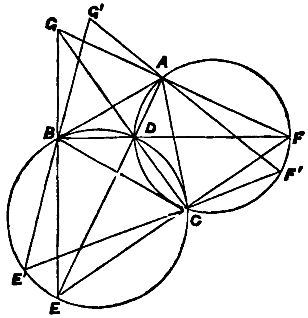
6. If n and l be positive integers, and $l =$ or $> \frac{1}{2}n$, to show that

$$\frac{(\lfloor n-l)^2}{\lfloor n-2l} = \lfloor n - l^2\lfloor n-1 + \frac{l^2(l-1)^2}{1\,.\,2}\lfloor n-2 - \text{\&c.}$$

The quantity on the left-hand side may easily be shown to be equal to the number of arrangements of n things in n compartments, subject to the condition that a particular l of them shall none of them be placed in any of l specified compartments; and the quantity on the right-hand side has already been shown equal to the same. It is not necessary that l should be a factor of n.

4742. (By the late W. Hopps.)—Through three given points to draw straight lines which, by their intersections in pairs, shall form the maximum triangle of given species.

Solution by Nilkanta Sarkar, B.A.



On AB, BC, AC draw segments containing angles supplementary to the given angles, and let two of them intersect in D; then the third also passes through their intersection, because the three given angles are equal to two right angles. Join AD, DC, BD, and draw FE, GE, FG perpendicular to DC, AD, BD respectively. These must meet in the circumference of the circles, because, DAF, DCF being each a right angle, GA and CE meet the diameter through D at F. Now any one triangle of the given species passing through A, C, F must have their vertices on the circumference of these circles. Let E′F′G′ be any such triangle. Now DF is a diameter, therefore DF > DF′, also DC > perpendicular from D on E′F′, therefore DFC > DF′C; similarly DEC > DE′C, hence EDF > E′DF′; similarly GDF > G′DF′ and GDE > G′DE′, therefore EFG > E′F′G′, therefore EFG is the maximum.

4708. (By R. F. Scott, B.A.)—If $P=0$, $Q=0$, $R=0$ be the equations to three planes meeting in a point A; prove that (1) the equation of the first polar of A with respect to any surface U is obtained by equating the Jacobian of P, Q, R, and U to zero; and (2) the equation of the second polar of A is obtained by bordering the Hessian of U by the coefficients of P, Q, and R, and equating the determinant so formed to zero.

Solution by the Rev. F. D. Thomson, M.A.

The equation of the first polar of the point (x', y', z', w') with respect to $U=0$, is
$$x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz} + w'\frac{dU}{dw} = 0.$$
We have also
$$lx' + my' + nz' + rw' = 0,$$
since (x', y', z', w') is on the plane $P=0$. Similarly
$$l'x' + m'y' + n'z' + r'w' = 0, \quad l''x' + m''y' + n''z' + r''w' = 0.$$
Eliminating x', y', z', w', and bearing in mind that $l = \frac{dP}{dx}$, &c., we have
$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz}, & \frac{dU}{dw} \\ l, & m, & n, & r \\ l', & m', & n', & r' \\ l'', & m'', & n'', & r'' \end{vmatrix} = 0, \text{ or } \begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz}, & \frac{dU}{dw} \\ \frac{dP}{dx}, & \frac{dP}{dy}, & \&c. & \\ \&c. \ldots\ldots & & & \end{vmatrix} = 0;$$
or $J=0$, when J is the Jacobian of U, P, Q, R.

Again, if $U'=0$ be the first polar of x', y', z', w' with respect to U, the polar of the same point with reference to U is the first polar with reference to U'; therefore, as above, its equation is
$$\begin{vmatrix} \frac{dU'}{dx}, & \frac{dU'}{dy}, & \frac{dU'}{dz}, & \frac{dU'}{dw} \\ l, & m, & n, & r \\ l', & m', & n', & r' \\ l'', & m'', & n'', & r'' \end{vmatrix} = 0.$$

But
$$\frac{dU'}{dx} = \frac{d}{dx}\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz}, & \frac{dU}{dw} \\ l, & m, & n, & r \\ l', & m', & n', & r' \\ l'', & m'', & n'', & r'' \end{vmatrix} = A\,(mn'r'') - G\,(nr'l'') + F\,(rl'm'') - H\,(lm'n'');$$
where
$$A = \frac{d^2U}{dx^2}, \quad G = \frac{d^2U}{dx\,dy}, \quad F = \frac{d^2U}{dz\,dx}, \quad H = \frac{d^2U}{dx\,dw};$$
and
$$(mn'r'') = \begin{vmatrix} m, & n, & r \\ m', & n', & r' \\ m'', & n'', & r'' \end{vmatrix} \&c.;$$
therefore
$$\frac{dU'}{dx} = \begin{vmatrix} A, & G, & F, & H \\ l, & m, & n, & r \\ l', & m', & n', & r' \\ l'', & m'', & n'', & r'' \end{vmatrix} \qquad \frac{dU'}{dy} = \begin{vmatrix} G, & B, & E, & K \\ l, & m, & n, & r \\ l', & m', & n', & r' \\ l'', & m'', & n'', & r'' \end{vmatrix}, \&c.$$

Proceeding in this way, we obtain

$$U'' \equiv \begin{vmatrix} A, & G, & F, & H, & l, & l', & l'' \\ G, & B, & E, & K, & m, & m', & m'' \\ F, & E, & C, & L, & n, & n', & n'' \\ H, & K, & L, & D, & r, & r', & r'' \\ l, & m, & n, & r, & 0, & 0, & 0 \\ l', & m', & n', & r', & 0, & 0, & 0 \\ l'', & m'', & n'', & r'', & 0, & 0, & 0 \end{vmatrix} = 0.$$

But $\begin{vmatrix} A, & G, & F, & H \\ G, & B, & E, & K \\ F, & E, & C, & L \\ H, & K, & L, & D \end{vmatrix} = 0$ is the Hessian.

4782. (By S. A. Renshaw.)—If MN be a variable chord of a conic such that (1) the sum of the distances of its extremities from one of the foci is constant, or (2) if the chord be drawn between opposite branches of an hyperbola, and the difference of the distances of its extremities from one of the foci remains constant; then prove that in both cases the locus of the middle point of the chord is a straight line, and in the first case is in fact one position of the variable chord itself.

Solution by R. F. Davis, B.A.

Let S be the focus; R the middle point of MN; and Mm, Rr, Nn perpendiculars on the corresponding directrix. Then

$$Rr = \tfrac{1}{2}(Mm \pm Nn) = \frac{1}{2e}(SM \pm SN) = \text{constant}.$$

Thus the locus of R is a straight line parallel to the directrix.

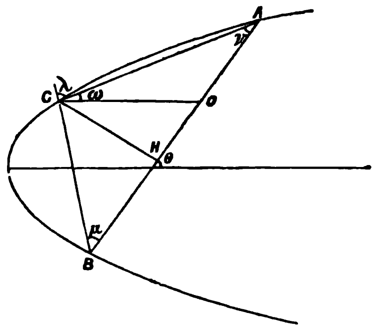
4776. (By Dr. Booth, F.R.S.)—A conic section is cut by a circle in four points, the vertices of an inscribed quadrilateral. Chords are drawn through a focus parallel to the sides and diagonals of this quadrilateral; prove that the square root of the continued product of the six focal chords is equal to the square of the diameter (D) of the circle multiplied by the parameter (p) of the conic section, or $p^2D^4 = f_1f_2f_3f_4f_5f_6$.

I. *Solution by the* EDITOR.

In Salmon's *Conics* (5th ed., pp. 202, 209, 321) it is shown that, if α, β, γ be the sides of a triangle inscribed in a conic; R the radius of the circumscribed circle; and f_1, f_2, f_3 focal chords parallel to α, β, γ respectively; then we have the relation $pD^2 = f_1 f_2 f_3$.

Now the six focal chords parallel to the sides and diagonals of the quadrilateral that is inscribed both in the circle and conic, constitute three pairs of equal chords; therefore calling these six chords $f_1, f_2 \dots f_6$, the above relation gives $p^2 D^4 = f_1 f_2 f_3 f_4 f_5 f_6$.

II. *Solution by the* PROPOSER.



If in a parabola a triangle be inscribed, the product of the three focal chords parallel to the sides of the triangle is equal to the product of the parameter p, into the square of the diameter (D) of the circumscribing circle, or

$$pD^2 = f_1 f_2 f_3.$$

For, let $CO = c$, $AO = a$, $BO = b$, $AB = l$, $BC = n$, $AC = m$, $CH = P$, a perpendicular on AB.

Now $\sin^2\nu : \sin^2\omega = c^2 : a^2$ and $\sin^2\omega : 1 = p : f_2$, therefore $\sin^2\nu : 1 = pc^2 : f_1 a^2$, for the same reason

$$\sin^2\mu : 1 = pc^2 : f_3 b^2;$$

and as $\sin^2\lambda : \sin^2\theta = l^2c^2 : n^2m^2$, and $\sin^2\theta : 1 = p : f_1$, therefore $\sin^2\lambda : 1 = pl^2c^2 : f_1^2 n^2 m^2$; hence $\dfrac{p^3 l^2 m^2 n^2 c^6}{\sin^2\lambda \sin^2\mu \sin^2\nu} = f_1 f_2 f_3 a^2 b^2 n^4 m^4$.

But $D^3 = \dfrac{lmn}{\sin\lambda \sin\mu \sin\nu}$, and $DP = mn$, therefore $p^3 c^6 D^2 = f_1 f_2 f_3 a^2 b^2 P^4$; also $P = c \sin\theta$, and $ab \sin^2\theta = pc$, therefore $pD^2 = f_1 f_2 f_3$.

Now, if a circle cut a parabola in four points, the quadrilateral whose vertices are at these points is equivalent to four inscribed triangles; hence, repeating the above construction and multiplying the relations, we shall have $p^4 D^8$ = product of twelve focal chords; but as each focal chord occurs twice, we shall have, taking the square root, $p^2 D^4 = f_1 f_2 f_3 f_4 f_5 f_6$.

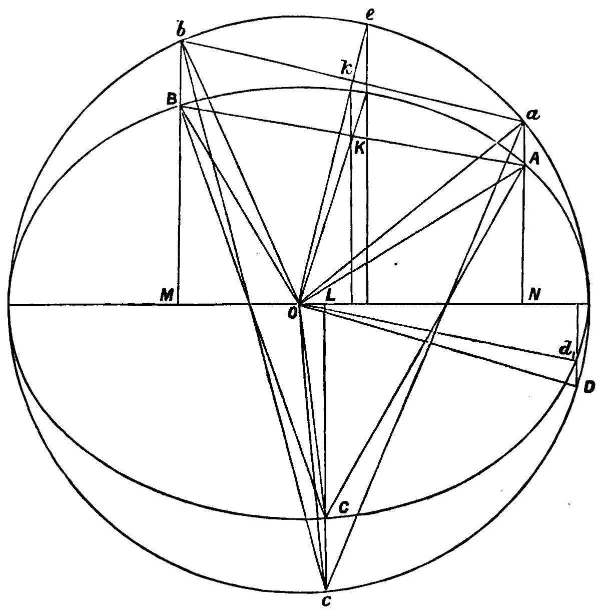
The same property may be found most simply for the ellipse by orthogonal projection. Let R be the radius of the circle circumscribing the triangle whose sides are α, β, γ inscribed in the ellipse whose semiaxes are a, b. Let this ellipse and inscribed triangle be projected into the circle whose radius is b, and into the triangle whose sides are $\alpha_1, \beta_1, \gamma_1$. Let S be the area of the triangle in the ellipse and S_1 its projection in the circle, then $S : S_1 = a : b$.

Now $4R = \dfrac{\alpha\beta\gamma}{S}$, and $4b = \dfrac{\alpha_1\beta_1\gamma_1}{S_1}$; and if d_1, d_2, d_3 be the semidiameters of the ellipse parallel to α, β, γ, we have $\dfrac{\alpha}{\alpha_1} = \dfrac{d_1}{b}$; hence $\dfrac{\alpha\beta\gamma}{\alpha_1\beta_1\gamma_1} = \dfrac{d_1 d_2 d_3}{b^3}$,

and $\dfrac{\alpha\beta\gamma}{\alpha_1\beta_1\gamma_1} = \dfrac{a\mathrm{R}}{b^2} = \dfrac{d_1 d_2 d_3}{b^2}$; therefore $ab\mathrm{R} = d_1 d_2 d_3$. The focal property follows at once from this; for $af_1 = 2d_1^2$, therefore $f_1 f_2 f_3 = 4p\mathrm{R}^2 = p\mathrm{D}^2$.

III. *Solution by* S. A. Renshaw.

Let ABC be a triangle inscribed in an ellipse. Draw AN, BM, CL



perpendicular to the axis, and produce them to meet the auxiliary circle in a, b, c respectively; bisect the arc ab in e; draw OD, od_1 parallel respectively to ab and AB; and complete the figure.

Put a and b for the semi-axes, and d_1 for the semi-diameter $\mathrm{O}d_1$,

then $a : b = b\mathrm{MN}a : \mathrm{BMNA} = \triangle\, b\mathrm{OM} : \triangle\,\mathrm{BOM} = \triangle\, a\mathrm{ON} : \triangle\,\mathrm{AON}$,

therefore $$\triangle\, b\mathrm{O}a : \mathrm{BOA} = a : b.$$

But $\triangle\, b\mathrm{O}a = \frac{1}{2}a^2 \sin b\mathrm{O}a$, therefore $\triangle\,\mathrm{BOA} = \frac{1}{2}ab \sin b\mathrm{O}a$,

similarly $\triangle\,\mathrm{BOC} = \frac{1}{2}ab \sin b\mathrm{O}c$, and $\triangle\,\mathrm{COA} = \frac{1}{2}ab \sin c\mathrm{O}a$;

therefore $$\triangle\,\mathrm{ABC} = \tfrac{1}{2}ab \left\{\sin b\mathrm{O}a + \sin b\mathrm{O}c - \sin(b\mathrm{O}a + b\mathrm{O}c)\right\}$$
$$= 2ab \sin \tfrac{1}{2}b\mathrm{O}a \sin \tfrac{1}{2}b\mathrm{O}c \sin \tfrac{1}{2}a\mathrm{O}c.$$

And since $ab : \mathrm{AB} = \mathrm{OD}\,(a) : \mathrm{O}d\,(d_1)$, and $ab = 2a \sin \frac{1}{2}b\mathrm{O}a$, therefore $\mathrm{AB} = 2d_1 \sin \frac{1}{2}b\mathrm{O}a$, similarly $\mathrm{BC} = 2d_2 \sin \frac{1}{2}b\mathrm{O}c$, and $\mathrm{CA} = 2d_3 \sin \frac{1}{2}c\mathrm{O}a$,

d_1, d_2, d_3 being semi-diameters parallel to AB, BC, CA respectively. Therefore, if R be the radius of the circle about ABC, we have

$$\mathrm{R} = \frac{\mathrm{AB}\,.\,\mathrm{BC}\,.\,\mathrm{CA}}{4\Delta\mathrm{ABC}} = \frac{d_1 d_2 d_3}{ab}, \text{ or } ab\mathrm{R} = d_1 d_2 d_3,$$

whence the rest follows as in Dr. BOOTH's solution.

4725. (By Prof. WOLSTENHOLME, M.A.)—Prove that

$$\int_0^{\frac{1}{2}\pi} (\cos x)^a \cos ax\,dx = \frac{\pi}{2^{a+1}} \quad \text{if } a > -1.$$

Solution by E. B. ELLIOTT, B.A.

The following addition to Bertrand's demonstration is sufficient.

Denoting $\int_0^{\frac{1}{2}\pi} (\cos x)^a \cos ax\,dx$ by u_a, we have

$$u_a = \int_0^{\frac{1}{2}\pi} (\cos x)^a \left\{\cos(a+1)\,x\cos x + \sin(a+1)\,x\sin x\right\} dx$$

$$= u_{a+1} + \int_0^{\frac{1}{2}\pi} (\cos x)^a \sin x \sin(a+1)\,x\,dx,$$

or, integrating by parts,

$$= u_{a+1} - \frac{1}{a+1}\left[(\cos x)^{a+1}\sin(a+1)\,x\right]_0^{\frac{1}{2}\pi} + u_{a+}\,.$$

Thus, if $a+1 > 0$, $\quad u_a = 2u_{a+1} = 2\,.\,\dfrac{\pi}{2^{a+2}},$

by Bertrand's result, $\quad = \dfrac{\pi}{2^{a+1}}.$

4821. (By Professor M. ROBERTS.)—Prove that

$$c\left(\frac{d^2y}{dx^2}\right)^{\frac{8}{3}} + c'\left(\frac{d^2y}{dx^2}\right)^{\frac{10}{3}} = \left(\frac{d^3y}{dx^3}\right)^2,$$

where c, c' are arbitrary constants, is a second integral of the differential equation of the fifth order which represents a conic section.

I. *Solution by* LIONEL H. ROSENTHAL.

Differentiating twice, we get the following two equations

$$4c\left(\frac{d^2y}{dx^2}\right)^{\frac{5}{3}} + 5c'\left(\frac{d^2y}{dx^2}\right)^{\frac{7}{3}} = 3\,\frac{d^4y}{dx^4} \quad \text{.....................} \; (1),$$

$$20c\left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}} + 35c'\left(\frac{d^2y}{dx^2}\right)^{\frac{4}{3}} = 9\,\frac{\frac{d^5y}{dx^5}}{\frac{d^3y}{dx^3}} \quad \text{..................} \; (2);$$

and combining these with the original equation, we can eliminate $c\left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}$ and $c'\left(\frac{d^2y}{dx^2}\right)^{\frac{4}{3}}$ linearly. The result is

$$\begin{vmatrix} \left(\frac{d^2y}{dx^2}\right)^2, & \left(\frac{d^2y}{dx^2}\right)^2, & \left(\frac{d^3y}{dx^3}\right)^2 \\ 4\left(\frac{d^2y}{dx^2}\right), & 5\left(\frac{d^2y}{dx^2}\right), & 3\left(\frac{d^4y}{dx^4}\right) \\ 20, & 35, & 9\,\frac{\left(\frac{d^5y}{dx^5}\right)}{\left(\frac{d^3y}{dx^3}\right)} \end{vmatrix} = 0,$$

which, when expanded, gives

$$9\left(\frac{d^2y}{dx^2}\right)^2 \cdot \frac{d^5y}{dx^5} - 45\,\frac{d^2y}{dx^2} \cdot \frac{d^3y}{dx^3} \cdot \frac{d^4y}{dx^4} + 40\left(\frac{d^3y}{dx^3}\right)^3 = 0,$$

the differential equation of a conic.

II. *Solution by* Professor WOLSTENHOLME.

The equation $\quad ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0$

gives us $\quad ax + hy + f + (hx + by + f)\,\frac{dy}{dx} = 0,$

whence $\quad \frac{d^2y}{dx^2}(hx + by + f) = -\left\{a + 2h\frac{dy}{dx} + b\left(\frac{dy}{dx}\right)^2\right\}$

$$= -\frac{\left\{a(hx + by + f)^2 - 2h(ax + hy + f)(hx + by + f) + b(ax + hy + f)^2\right\}}{(hx + by + f)^3}$$

$$= \frac{(h^2 - ab)(ax^2 + by^2 + 2fy + 2gx + 2hxy) + 2fgh - af^2 - bg^2}{(hx + by + f)^3}$$

$$= \frac{\Delta}{(hx + by + f)^3}, \text{ or } \frac{d^2y}{dx^2} = \frac{\Delta}{(hx + by + f)^3},$$

whence $\dfrac{d^3y}{dx^3} = -\dfrac{3\Delta\left(h + b\dfrac{dy}{dx}\right)}{(hx+by+f)^4} = -\dfrac{3\Delta\left\{h(hx+by+f) - b(ax+hy+f)\right\}}{(hx+by+f)^5}$

$$= \frac{3\Delta\left\{(ab-h^2)x + bg - hf\right\}}{(hx+by+f)^5},$$

or $$\left(\frac{d^3y}{dx^3}\right)^2 = \frac{9\Delta^2\left\{(ab-h^2)x + bg - hf\right\}^2}{(hx+by+f)^{10}}.$$

Now $$(hx+by+f)^2 = (h^2-ab)x^2 + 2(hf-bg)x + f^2 - bc$$

$$= \frac{\left\{(ab-h^2)x + bg - hf\right\}^2 - b\Delta}{h^2-ab},$$

whence $$\left(\frac{d^3y}{dx^3}\right)^2 = \frac{9\Delta^2}{(hx+by+f)^{10}}\left\{b\Delta + (h^2-ab)(hx+by+f)^2\right\}$$

$$= \frac{9b}{\Delta^{\frac{1}{3}}}\left(\frac{d^2y}{dx^2}\right)^{\frac{10}{3}} + \frac{9(h^2-ab)}{\Delta^{\frac{2}{3}}}\cdot\left(\frac{d^2y}{dx^2}\right)^{\frac{8}{3}},$$

so that one second integral of the general differential equation of a conic must be $$\left(\frac{d^3y}{dx^3}\right)^2 = C\left(\frac{d^2y}{dx^2}\right)^{\frac{10}{3}} + C'\left(\frac{d^2y}{dx^2}\right)^{\frac{8}{3}},$$

where C, C′ are arbitrary constants. Eliminating C and C′ from this equation is probably the easiest way of finding the general differential equation $$9\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5} = 5\frac{d^3y}{dx^3}\left\{9\frac{d^2y}{dx^2}\frac{d^4y}{dx^4} - 8\left(\frac{d^3y}{dx^3}\right)^2\right\}$$ of any conic.

4822. (By Professor CROFTON.)—Two straight lines AB, CD have a common part CB; prove that the mean distance between two points, one taken at random in AB, the other in CD, is $M = \dfrac{AD^3 + CB^3 - AC^3 - BD^3}{6AB\,.\,CD}$.

I. *Solution by* S. FORDE, M.A.

Let AB = b, AC = c, AD = d; and let X and Y be the two points taken at random in AB, CD respectively; AX = x, AY = y. Then three cases may occur:—

(1). X may lie in AC; then Y may lie in CD, and XY = $y - x$.
(2). X may lie in CB, Y in CX; then XY = $x - y$.
(3). X may lie in CB, Y in XD; then XY = $y - x$.

Hence the mean value of XY =

$$\int_0^c\int_c^d (y-x)\,dx\,dy+\int_c^b\int_c^x (x-y)\,dx\,dy+\int_c^b\int_x^d (y-x)\,dx\,dy \div \int_0^b\int_c^d dx\,dy$$

$$=\left\{\tfrac{1}{2}cd\,(d-c)+\tfrac{1}{3}(b-c)^3+\tfrac{1}{3}(b-d)^3-(c-d)^3\right\}\div b\,(d-c),$$

which gives the result stated in the Question.

II. *Solution by the* PROPOSER.

The whole number of cases, of which we have to find the mean, is (if we suppose the random point X to range over AB, Y over CD, *and* also Y over AB, X over CD) 2AB . CD. But these cases are the same as when X and Y range over the whole line AD, leaving out the cases when both are in AC, and when both are in BD, and *repeating* the cases when both are in CB. Now, as the sum of n quantities is n times the mean, we have [if we write M (AD) for "the mean distance of two points in AD"]

$$2AB.CD.M=AD^2.M\,(AD)+CB^2.M\,(CB)-AC^2.M\,(AC)-BD^2.M\,(BD)$$

But the mean distance between two points in a line is $\frac{1}{3}$ of the line,

therefore $$M=\frac{AD^3+CB^3-AC^3-BD^3}{6AB\,.\,CD}.$$

III. *Solution by* E. B. ELLIOTT, B.A.

Let x and y be the distances of the two points from A; then we have

$$M=\left\{\int_0^{AC}dx\int_{AC}^{AD}(y-x)\,dy+\int_{AC}^{AB}dx\int_x^{AD}(y-x)\,dy+\int_{AC}^{AB}dx\int_{AC}^{x}(x-y)\,dy\right\}\div AB\,.\,CD$$

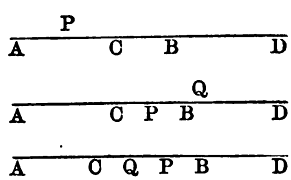
$$=\left\{\tfrac{1}{2}(AD^2-AC^2)\,AC-\tfrac{1}{2}(AD-AC)\,AC^2-\tfrac{1}{3}(AD-AB)^3+\tfrac{1}{3}(AD-AC)^3+\tfrac{1}{3}(AB-AC)^3\right\}\div AB\,.\,CD$$

$$=\frac{3AD^2\,.\,AC-3AD\,.\,AC^2+(AD-AC)^3-BD^3+CB^3}{6AB\,.\,CD}$$

$$=\frac{AD^3-AC^3-BD^3+CB^3}{6AB\,.\,CD}.^{*}$$

IV. *Solution by* Professor WOLSTENHOLME, M.A.

If AB = a, CD = b, and CB = c, the mean value will be,



$$\frac{1}{ab}\left\{\int_0^b\int_0^{a-c}(x+y)\,dx\,dy+\int_0^c\int_x^b (y-x)\,dy\,dx+\int_0^c\int_0^x (x-y)\,dy\,dx\right\},$$

[* Because, if X, Y are taken at random in a line, the chance that a third point Z, taken at random, will fall on XY, is evidently $\frac{1}{3}$; as one of the three must be the middle one.]

the three integrals answering to the cases (1) when P is in AC, (2) when P is in CB and Q in PD, (3) when P is in CB and Q in CP. This gives

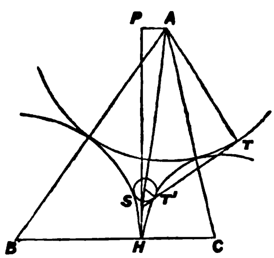
$$\frac{1}{ab}\left[\frac{(a-c)b(a-c+b)}{2}+\tfrac{1}{6}\left\{b^3-(b-c)^3\right\}+\tfrac{1}{6}c^3\right]$$

or
$$\frac{1}{6ab}\left[\left\{(a-c+b)^3-(a-c)^3-b^3\right\}+\left\{b^3-(b-c)^3\right\}+c^3\right],$$

or
$$\frac{1}{6\,\overline{AB}\,.\,\overline{CD}}\left(AD^3+CB^3-AC^3-BD^3\right).$$

4690. (By A. MARTIN.)—Three circles, of radii a, b, c, touch each other externally; find the radii of three circles drawn in the space enclosed by them, each touching the other two and two of the given circles.

Solution by ASHER B. EVANS, M.A.



If two circles, of radii r_1, r_2, touch externally a third circle of radius R, the length of the line joining the two points of contact

is
$$\frac{R(12)}{\left\{(R+r_1)(R+r_2)\right\}^{\frac{1}{2}}}$$

where (12) is the length of a common tangent to the circles r_1, r_2.

If four circles, radii r_1, r_2, r_3, r_4, touch externally a common fifth circle of radius R, the four points of contact form a quadrilateral inscribed in the fifth circle. The product of the diagonals of this quadrilateral being equal to the sum of the products of the opposite sides, we have, on suppressing the common factor $R^2 \div \left\{(R+r_1)(R+r_2)(R+r_3)(R+r_4)\right\}^{\frac{1}{2}}$,

$$(12)(34)+(23)(14) = (13)(24)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If each of the four circles r_1, r_2, r_3, r_4 be tangential to two of the other three, equation (1) gives $8\,(r_1r_2r_3r_4)^{\frac{1}{2}} = (13)(24)$ .... (2).

Let A, B, C, D, E, F be the centres, and a, b, c, d, e, f the radii, of the six circles in the question; also let t_1, t_2, t_3 be common tangents to A and D, B and E, C and F; then, from equation (2), we have

$$8\,(abde)^{\frac{1}{2}} = t_1t_2,\quad 8\,(bcef)^{\frac{1}{2}} = t_2t_3,\quad 8\,(acdf)^{\frac{1}{2}} = t_3t_1;$$

whence $\quad t_1 = 2\,(2ad)^{\frac{1}{2}},\quad t_2 = 2\,(2be)^{\frac{1}{2}},\quad t_3 = 2\,(2cf)^{\frac{1}{2}}$ .... (3).

Let S be the external centre of similitude of A and D, then S will be in the radical axis HP of B and C; for A and D both touch B and C.

Let M, N be the points in which A and D touch C; then, since M, N are anti-homologous points, MN passes through S; and since TT′ also

passes through S, we have $SH^2 = SM \cdot SN = ST \cdot ST'$ .... (4).

From the similar triangles ST'D and STA, we have

$$\frac{ST'}{d} = \frac{ST}{a} = \frac{T'T}{a-d} = \frac{t_1}{a-d}; \text{ therefore } ST' = \frac{dt_1}{a-d}, \quad ST = \frac{at_1}{a-d},$$

$$SH = \frac{t_1 (ad)^{\frac{1}{2}}}{a-d} = \frac{2ad\sqrt{2}}{a-d}.$$

From the geometry of the figure, we have, furthermore,

$$PH = \frac{2}{b+c}\left\{abc(a+b+c)\right\}^{\frac{1}{2}}, \quad AP = a\left(\frac{b-c}{b+c}\right);$$

hence $\qquad PS = PH - SH = (ST^2 + AT^2 - AP^2)^{\frac{1}{2}}$

gives on reduction, and by analogy, for the three radii,

$$\frac{1}{d} = \frac{1}{a} + \frac{2}{b} + \frac{2}{c} + 2\left(\frac{2}{ab} + \frac{2}{ac} + \frac{2}{bc}\right)^{\frac{1}{2}},$$

$$\frac{1}{e} = \frac{2}{a} + \frac{1}{b} + \frac{2}{c} + 2\left(\frac{2}{ab} + \frac{2}{ac} + \frac{2}{bc}\right)^{\frac{1}{2}},$$

$$\frac{1}{f} = \frac{2}{a} + \frac{2}{b} + \frac{1}{c} + 2\left(\frac{2}{ab} + \frac{2}{ac} + \frac{2}{bc}\right)^{\frac{1}{2}}.$$

If the radicals in the last three expressions be taken negatively, we shall have the reciprocals of the radii of three circles *enclosing* the circles A, B, C.

4562. (By S. Bills.)—If R denote the integral value of x which will make $Ax^2 + 1 =$ a square; and r the least integral value of x which will make $Ax^2 - 1 =$ a square (when this is possible); prove that R will always be a *multiple* of r; that is, we shall have $R = mr$, m being a whole number.

I. *Solution by* Artemas Martin.

Put $\qquad Ax^2 + 1 = y^2, \quad Ax^2 - 1 = z^2$ ....(1, 2).

Let $p =$ the value of y corresponding to $x = R$, and $q =$ the value of z corresponding to $x = r$; then we have

$$p^2 - AR^2 = 1, \quad q^2 - Ar^2 = -1 \quad \text{..................(3, 4)};$$

which may be written

$$(p - RA^{\frac{1}{2}})(p + RA^{\frac{1}{2}}) = 1, \quad (q - rA^{\frac{1}{2}})(q + rA^{\frac{1}{2}}) = -1 \text{ ...(5, 6)}.$$

Raising (6) to the $2n$th power, we have

$$(q - rA^{\frac{1}{2}})^{2n}(q + rA^{\frac{1}{2}})^{2n} = 1 \text{ ....(7)}.$$

Since R represents any integral value of x satisfying (1), we may take

$$p + RA^{\frac{1}{2}} = (q + rA^{\frac{1}{2}})^{2n}, \quad p - rA^{\frac{1}{2}} = (q - rA^{\frac{1}{2}})^{2n};$$

whence $\qquad R = \dfrac{(q + rA^{\frac{1}{2}})^{2n} - (q - rA^{\frac{1}{2}})^{2n}}{2A^{\frac{1}{2}}}.$

Expanding and dividing throughout by $2A^{\frac{1}{2}}$, we obtain

$$R = 2nq^{2n-1}r + \frac{2n(2n-1)(2n-2)}{1.2.3}q^{2n-3}r^3A$$

$$+ \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)}{1.2.3.4.5}q^{2n-5}r^5A^2 + \dots + 2nqr^{2n-1}A^{n-1},$$

$$= r\left[2nq^{2n-1} + \frac{2n(2n-1)(2n-2)}{1.2.3}q^{2n-3}r^2A\right.$$

$$\left. + \frac{2n(2n-1)(2n-2)(2n-3)\,2n-4)}{1.2.3.4.5}q^{2n-5}r^4A^2 + \dots + 2nqr^{2n-2}A^{n-1}\right].$$

II. *Solution by* DR. HART.

Put y for x; then $Ay^2+1 = \square = x^2$, and $Ay^2-1 = \square = x^2$;
and we must have $x^2-Ay^2 = 1$, and $x^2-Ay^2 = 1$ ....(1, 2).
Now (Barlow's *Theory of Numbers*, pp. 383, 384) if p, q are the least values of x, y in (2), the next values of x, y are p^2+Aq^2 and $2pq$. The three forms given by Barlow may be reduced to two, viz., $x^2-Ay^2 = 1$ and $x^2-Ay^2 = \pm 1$. If $R = 2pq$, $r = q$, then $m = 2p$. This relation will only hold good in the second formula; for, if p, q be the least values of x, y in the first formula, q has not the same value as q in the other, and is sometimes incommensurable with it, as will be seen from the Tables. In my Solution of 2981, on pp. 63, 64, Vol. XX., *Reprint*, I consider $x = 1$, $y = 0$ as the first values; r, s the second values; and $2r^2+1$, $2rs$ the third values, which correspond to Barlow's second values; and the relation of $2rs$ to s is the same as in his.

III. *Solution by* ASHER B. EVANS, M.A.

Lagrange has shown that the condition $y^2-Ax^2 = -1$ is possible in integers only when the last figure of the first period of quotients found from the extraction of the square root of A, after the manner of continued fractions, occupies an odd place in that period; and that the least values of y and x that will satisfy this condition are the numerator and denominator of the converging fraction corresponding to said last figure. He has also shown that, when the first period consists of an odd number of figures, the least values of y and x that will satisfy condition $y^2-Ax^2 = 1$ in integers, are the numerator and denominator of the converging fraction corresponding to the last figure of the second period of quotients.

Let a be the nearest integer to $A^{\frac{1}{2}}$, smaller than $A^{\frac{1}{2}}$, and let $a_1, a_2, a_3 \dots a_n$ be the n quotients of the first period; then the denominators of the n converging fractions corresponding to these quotients being $d_1, d_2, d_3 \dots d_n$, we have $d_1 = 1$, $d_2 = a_1$, $d_3 = a_1 a_2 + 1 \dots d_{n-1} = N$ (where N may be calculated when n is known), $d_n = r$. Let $\delta_1, \delta_2, \delta_3 \dots \delta_n$ represent the denominators of the n converging fractions of the second period; then $M_1, M_2, M_3 \dots M_n$ being multiples of r, we have $\delta_1 = a_n r + N = M_1 + Nd_1$, $\delta_2 = a_1 a_n r + a_1 N + r = M_2 + Na_1 = M_2 + Nd_2$, $\delta_3 = M_3 + Nd_3 \dots\dots \delta_n = M_n + Nd_n = Mn + Nr =$ a multiple of r.

Hence the least integral value of x that will make $Ax^2-1=\square$ is an exact divisor of the *least* value of x that will make $Ax^2+1=\square$, and therefore of *every* value of x that will make $Ax^2+1=\square$. Therefore $R=mr$.

IV. *Note by the* PROPOSER.

Since sending my solution to this Question (published in the *Reprint*, Vol. XXIII., p. 109), I have discovered the following very simple and general relations in reference thereto. Suppose we have

$$x^2-Ay^2=-1 \text{ and } u^2-Av^2=+1 \quad\text{..................(1, 2)},$$

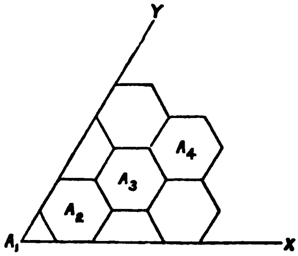
and suppose x and y to be *any* numbers that satisfy (1); then, if we take $u=2x^2+1$ and $v=2xy$, (2) will be satisfied; and, if x and y be the *least* numbers that will satisfy (1), then $u=x^2+1$, and $v=2xy$ will be the *least* numbers that will satisfy (2), and v will be a multiple of y.

Again, if $\quad x^2-Ay^2=+1$ and $u^2-Av^2=+1$ ....(3, 4); and if x and y be *any* numbers that satisfy (3), then if we take $u=2x^2-1$, and $v=2xy$, (4) will be satisfied; so that, knowing one solution of (3), we may thus readily obtain others. These elegant relations may be very easily proved as follows:—In (2) substitute $2x^2+1$ for u and $2xy$ for v; we have $(2x^2+1)^2-4Ax^2y^2=1$, or $4x^2(x^2+1-Ay^2)=0$; whence $x^2-Ay^2=-1$, or x and y must satisfy (1).

Again, in (4) substitute $2x^2-1$ for u, and $2xy$ for v; then $(2x^2-1)^2-4Ax^2y^2=1$, or $4ux^2(x^2-1-Ay^2)=0$; whence $x^2-Ay^2=+1$, or x and y must satisfy (3).

4867. (By the Rev. Dr. WILSON.)—A lady lately amused herself by making a piece of patchwork, each patch being a regular hexagon. Supposing that there are n rows, the central hexagon being reckoned as one row; show that the number of patches is $1-3n+3n^2$.

I. *Solution by* CHRISTINE LADD, Rev. G. H. HOPKINS, M.A., *and others.*



Every regular hexagon can exactly be surrounded by six equal regular hexagons.

If A_1 be the centre of the central hexagon, A_1X, A_1Y straight lines drawn through two of its angular points, which lie at the extremities of a side, the angle YA_1X will just contain A_2, a hexagon of the second row.

A_2 is surrounded by the central hexagon A_1, two hexagons of the second row, and three of the third; of these last the hexagon A_3 and two halves lie in the angle YA_1X; hence in the third row there will be twelve hexagons.

The three hexagons of the fourth row, which lie on sides of A_3 are just included in the angle YA_1X; hence there are eighteen in the fourth row. It can easily be seen that three complete hexagons and two halves of the fifth row lie in the angle YA_1X, which gives twenty-four hexagons in the fifth row.

Hence the whole number in the work will be the sum of the series

$$1+6+12+18+24, \text{\&c., or } 1+6\left\{1+2+3+\ldots\ldots n-1\right\};$$

that is, $$1+6\times\tfrac{1}{2}n(n-1), \text{ or } 1+3n^2-3n.$$

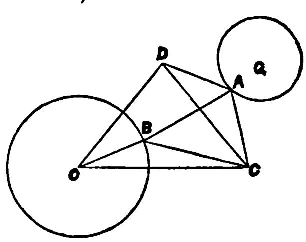
II. *Solution by* R. TUCKER, M.A., A. B. EVANS, M.A., *and others.*

The question is the same as that of finding the number of equal circles which can be placed round an equal circle, and so on, for the hexagons can be readily described round each circle. Drawing the sextantal lines, we see that each row after the second has one more circle in each sextantal space than its predecessor; hence the nth row has 6 more circles, or hexagons, than the $(n-1)$th; hence the whole number required is

$$1+\tfrac{1}{2}(n-1)\left\{12+6(n-2)\right\} = 1-3n+3n^2.$$

4799. (By the EDITOR.)—Construct a triangle of given species, which shall have a vertex on each of two given circles, and the third at a given point.

Solution by H. MURPHY, E. RUTTER, *and others.*



Suppose ABC to be the required triangle, having two of its vertices A, B on the circles (Q), (O), and the third vertex at the given point C. Join OC, and on it make a triangle OCD of the given species. Then, since $\angle OCB = DCA$, and $OC : CB = DC : CA$, the triangles OCB, DCA are similar (Euc. VI. 6); hence $CO : OB = CD : DA$; therefore DA is given, and consequently the point A is determined by drawing a circle around D with radius DA.

5 6

Printed by C. F. Hodgson & Son, Gough Square, Fleet Street, E.C.